RESEARCH STATEMENT

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My research interests lie in the area of *geometric group theory*, which aims to understand finitely presented groups via their intrinsic geometric and topological properties. More specifically, I focus on the study of *random groups* in the density model introduced by M. Gromov in [Gro93, §9]. I am also interested in A. Zuk's triangular presentation model of random groups [Żuk03] and Gromov's expander graph model of random groups [Gro03].

In general, a random group can be defined as a random variable with values in a given set of finitely presented groups (cf. [Gro03, §1.9], [Tsa22b, Defnition 1]). There are two main objectives in the study of random groups: exhibitions of generic properties for finitely presented groups, such as G. Arzhantseva and A. Ol'shanskii's result in [AO96]; and constructions of exotic examples of groups with surprising properties, such as the Gromov monster group introduced in [Gro03].

In the density model, a random group is defined by a finite presentation where the set of generators is fixed and a *density parameter* $d \in [0, 1]$ is given to determine the number of randomly chosen relators. The main concerns of the density model are the *phase transitions*: when the density d passes through some critical value, certain properties of the random group change dramatically. The first example is in [Gro93, §9.B], at density $d = 1/2$: when d passes through $1/2$, the random group goes from a non-elementary hyperbolic group of cohomological dimension 2, to a trivial group.

In [Gro03, §1.9], Gromov gave a list of problems concerning phase transitions in (more generally defined) random groups. In particular, the problem (iv) asks for "*existence/nonexistence of non-free subgroups*", which inspired one of the main topics in my research ([Tsa21b], §2).

The sections §1, §2 and §3 concern the results of my Ph.D. thesis [Tsa22b], which are respectively the intersection formula, a phase transition for free subgroups (the Freiheitssatz), and a phase transition for van Kampen diagrams. In §2 and §3, further goals to complete the projects are presented. The sections §4 and §5 are two new research projects, which concern respectively the parallel geodesics in random groups and the study of the expander graph model. Roughly speaking, there are currently four research directions, listed below.

- Complete the strong Freiheitssatz phase transition: Prove or disprove that for a random group with m generators at density $d < d_r = \min\{1/2, 1 - \log_{2m-1}(2r - 1)\}$, *every* subgroup of rank r is free (§2).

- Seek applications for the phase transition for the existence of van Kampen 2-complexes. That is, construct non-planar van Kampen 2-complexes in random groups that provide "interesting" group properties (§3).

- Study intrinsic geometric properties of random groups. More precisely, for a random group at density d, express (or estimate) the maximal number of parallel geodesics and the injectivity radius of its Cayley graph as functions of the density d (§4).

- Study the expander graph model of random groups. Prove or disprove the two speculated phase transitions, one for the $C'(\lambda)$ graphical small cancellation condition, and the other for hyperbolicity-triviality (§5).

§1 The intersection formula for random subsets

The *density* of a subset A in a finite set E is defined by $\operatorname{dens} A := \log_{|E|}(|A|)$. The *intersection formula* for random subsets is stated by M. Gromov in [Gro93, §9]: "Random subsets" A and B of a finite set E satisfy $\operatorname{dens}(A \cap B) = \operatorname{dens} A + \operatorname{dens} B - 1$. It is an essential probabilistic tool to deal with phase transitions in random groups. However, neither a precise definition of a random subset with density nor a detailed proof of the intersection formula was given by the author.

My first contribution in the area of random groups is the article [Tsa21a], in which we established a general framework for the study of random subsets with densities, and proved the intersection formula for the class of random subsets that are *densable* and *permutation invariant* (defined in [Gro93, p.272]). The same formula for the intersection between a random subset and a fixed subset is also proved.

Moreover, we established the *multidimensional intersection formula*: For any integer $k \geq 2$, if X is a fixed subset of E^k satisfying the d-small intersection condition introduced in [Tsa21a, §3.1] and R is a random subset of a finite set E and , then the intersection formula still holds: $\operatorname{dens}(X \cap R^k) = \operatorname{dens} X + \operatorname{dens} R^k - 1$. This result is useful for proving the phase transition of the $C'(\lambda)$ small cancellation condition in random groups [Tsa21a, Theorem 3], and is the key to proving our phase transition for the existence of van Kampen 2-complexes [Tsa22a, Theorem 1.5].

§2 Free subgroups: the Freiheitssatz in random groups

The study of free subgroups in finitely generated groups is a classical subject in combinatorial and geometric group theory.

The *Freiheitssatz* (*freedom theorem* in German) is a fundamental theorem in this field, proposed by M. Dehn and proved by W. Magnus in his doctoral thesis [Mag30] in 1930 (see [LS77, Ch. II.5]). The theorem states that for a group presentation with m generators and one cyclically reduced relator, if the last generator appears in the single relator, then the first $m-1$ generators freely generate a free subgroup. In [AO96], G. Arzhantseva and A. Ol'shanskii proved a strong version of the Freiheitssatz in few-relator random groups: *every* $(m-1)$*-generated subgroup of a* m*-generated few-relator random group is free*. As an application of the intersection formula, it was shown in [Tsa21a] that the "every $(m-1)$-generated subgroup is free" property holds for m-generated random groups at density $d < \frac{1}{120m^2\ln(2m)}$, while this density is not big enough to achieve a phase transition.

The phase transition for the Freiheitssatz in random groups. We say that a group presentation with m generators satisfies the *Freiheitssatz property* if the first $m-1$ generators freely generate a free subgroup. The main result of my second article [Tsa21b] is to highlight a new phase transition phenomenon for the Freiheitssatz property in random groups. More generally, for any integer r between 1 and $m-1$, there is a phase transition: the r first generators generate either a free subgroup or the whole group.

Theorem 1 (T., [Tsa21b, Theorem 1]). *Let $m \geq 2$ and $1 \leq r \leq m-1$. Let G be a random group with m generators at density d. There is a phase transition at density*

$$d_r = \min\left\{\frac{1}{2}, 1 - \log_{2m-1}(2r-1)\right\}.$$

(i) If $d < d_r$, then the first r generators generate a free subgroup of G.

(ii) If $d > d_r$, then the first r generators generate a the whole G.

There is an interesting corollary.

Corollary 2 (T., [Tsa21b, Corollary 2]). *If $d_r < d < d_{r-1}$, then the random group G admits a presentation with r generators (and the same number of relators) satisfying the Freiheitssatz property.*

The "strong" Freiheitssatz property. Inspired by Arzhantseva-Ol'shanskii's work [AO96], although only *"the first r generators"* is stated in the case $d < d_r$ of Theorem 1, the freeness may hold for *"every r-generated subgroup"*. The further goal of completing the project is to answer the following question.

Question 3. *Is d_r the critical density for the property "every r-generated subgroup is free"?*

If the answer to this question is positive, then *a random group with m generators at density $d_r < d < d_{r-1}$ is of rank r.*

In [Tsa21b], we proved that when $d < d_r$, every subgroup generated by a reduced X-labeled graph Γ with $b_1(\Gamma) \leq r$ and $|\Gamma| \leq \frac{d_r - d}{5}\ell$ (much smaller than the hyperbolicity constant) is free. An early work of T. Delzant [Del91] provides the strategy to study the freeness of subgroups generated by words much longer than the hyperbolicity constant (see also [Gro87, 5.3.A]). It remains the subgroups generated by r words of intermediate length, but it seems to be the most difficult part. Mixing the three scales of subgroups would be a sticky but feasible work.

§3 Existence of Van Kampen 2-complexes in random groups

The notion of a van Kampen diagram was introduced by E. van Kampen in [Kam33] to characterize if a particular word in the generators of a group given by a group presentation represents the identity element in that group. To prove the hyperbolicity of a random group at density $d < 1/2$ in [Gro93, 9.B], Gromov showed that with high probability, every bounded and reduced van Kampen diagram D of a random group at density d satisfies the isoperimetric inequality $|\partial D| \geq (1 - 2d - s)|D|\ell$. The non-reduced van Kampen 2-complex version of this inequality was thoroughly proved in [Tsa22a, Chapter 2], adapting the result of D. Gruber and J. Mackay in [GM18, §2] for random triangular groups.

The main results in the preprint [Tsa22a] is to highlight the phase transition for the existence of van Kampen 2-complexes with a given *geometric form* ([Tsa22a, Definition 3.1]) in random groups.

The phase transition. We say that a 2-complex Y is fillable by a group presentation G if there exists a reduced van Kampen 2-complex of G whose underlying 2-complex is Y. The fillability of 2-complex Y is characterized by its *critical density* $\mathrm{dens}_c(Y)$ ([Tsa22a, Definition 3.2]), and the phase transition happens at density $d = 1 - \mathrm{dens}_c(Y)$.

Theorem 4 (T., [Tsa22a, Theorem 1.5]). *Let Y be a 2-complex with a given geometric form. Let G be a random group with m generators at density d.*

(i) If $d < 1 - \mathrm{dens}_c Y$, then with high probability Y is not fillable by G.

(ii) If $d > 1 - \mathrm{dens}_c Y$ and Y is fillable by the set of all relators, then with high probability Y is fillable by G.

The first assertion of this theorem implies Gromov's isoperimetric inequality in random groups; and the second assertion implies the *converse*, as given in the following corollary.

Corollary 5 (T., [Tsa22a, Corollary 1.7])**.** *Let G be a random group with m generators at density d, with maximal relator length ℓ. Let $s > 0$ and $K > 0$. Let D be a finite planar 2-complex. If every sub-2-complex D' of D satisfies*

$$|\partial D'| \geq (1 - 2d + s)|D'|\ell,$$

then with high probability, D is fillable by G.

Applications. As an application of this result, we showed that there is a phase transition for the $C(p)$ small cancellation condition ([Tsa22a, Proposition 4.2]): If $d < 1/(p+1)$, then the $C(p)$ condition is satisfied; while if $d > 1/(p+1)$, then the $C(p)$ condition is not satisfied.

An analog of Corollary 5 for non-planar 2-complexes can also be established. Unfortunately, as fillable 2-complexes in a random group at density $d < 1/2$ are all contractible to a graph ([Tsa22a, Proposition 2.11]), no interesting application has been found.

Project 6. *Seek interesting group presentation properties that can be characterized by certain non-planar and contractible van Kampen 2-complexes with large enough critical densities (at least $1/2$), and apply them in random groups.*

§4 Geometric properties in random groups: parallel geodesics and injectivity radius

The aim is to study geometric properties of random groups at densities $d < 1/2$ (which are hyperbolic groups). One of the objectives is to adapt Gruber-Mackay's triangular random group results in [GM18] to the Gromov density model. The first steps are to estimate the maximal number of parallel geodesics and the injectivity radius, as functions of the density d.

Parallel geodesics. For a hyperbolic group G, denote as $P(G)$ the maximal number k such that k geodesics can be parallel in its Cayley graph. If δ is the hyperbolicity constant, then there exit C and a such that $P(G)$ is bounded by $C\exp(a\delta)$ (c.f. [CDP90, Chapitre 2]). Although not explicitly written, the main technical point in [GM18, §3] is to show that for a random triangular group G at density $d < \frac{11-\sqrt{41}}{12}$, there is a much smaller upper bound of $P(G)$ depending only on the density d.

By similar arguments, the number of parallel geodesics in a regular random group at density $d < 1/4$ can be bounded by a number that depends only on the density d.

Proposition 7. *Let G be a random group at density d. If $d < \frac{1}{4}$, then with high probability*

$$P(G) \leq 2 + \frac{2d}{1-4d} =: k(d).$$

Since the number $k(d)$ *diverges* as the density d increases to $1/4$, when $d > 1/4$, the maximal number of parallel geodesics may no longer be uniformly bounded (i.e. $P(G)$ may increase with the relator length ℓ), and there may be a *phase transition* at density $1/4$.

Guess 8. *Let G be a random group at density d. If $d > \frac{1}{4}$, then for any integer k, with high probability $P(G) > k$.*

Injectivity radius. The *injectivity radius* of a hyperbolic group G (acting on its Cayley graph) is the infimum of the stable lengths of its loxodromic elements (cf. [Cou16, Definition 3.34]). According to the arguments by T. Delzant in [Del96, Proposition 3.1], the stable lengths are at least $1/P(G)$. As a consequence of Proposition 7, the injectivity radius of a random group at density $d < 1/4$ is at least $1/k(d)$. Moreover, we have the following guess.

Guess 9. *Let G be a random group at density d. If $d < 1/4$, then with high probability the injectivity radius of G is* 1.

On the other hand, at density $d > 1/4$, as the number of parallel geodesics $P(G)$ may diverge according to Guess 8, the injectivity radius may converge to 0.

Guess 10. *Let G be a random group at density d. If $d > 1/4$, then for any $s > 0$, with high probability the injectivity radius of G is smaller than s.*

The Burnside problem in random groups. The main result of [GM18] concerns the Burnside problem in triangular random groups. They showed that at small enough densities, for n large enough, the n-periodic Burnside quotient of a random triangular group is infinite.

Theorem 11 (Gruber-Mackay, [GM18, Theorem 1.2])**.** *Denote $G(n, d)$ as the n-periodic Burnside quotient of a random triangular at density d. For any $d_0 < \frac{11-\sqrt{41}}{12} \approx 0.38307$, there exists an integer $n_0 \in \mathbb{N}$ such that for any $0 < d \leq d_0$ and any $n \geq n_0$, with high probability the random group $G(n, d)$ is infinite.*

This theorem is proved by establishing uniform bounds on *acylindricity constants* of triangular random groups, where the maximal number of parallel geodesics is involved (cf. [GM18, Theorem 1.4], see also [Cou16, Proposition 6.1]). We may ask the same question in the Gromov density model of random groups.

Question 12. *For any $d_0 < 1/2$, does there exist $n_0 \in \mathbb{N}$ such that for any $0 < d \leq d_0$ and any $n \geq n_0$, with high probability the n-periodic Burnside quotient of a random group at density d is infinite?*

Proposition 7 is not enough to answer this question, since there is an essential difference between the triangular density model and the Gromov density model: The hyperbolicity constant of (the Cayley graph of) a random triangular group at density $d < 1/2$ is bounded by $\frac{12}{1-2d}$, depending only on the density d. While the hyperbolicity constant for a regular random group at density $d < 1/2$ is $\frac{4\ell}{1-2d}$, which grows with the maximal relator length ℓ.

§5 The expander graph model of random groups

In the seminal article *"Random walk in random groups"* [Gro03], Gromov constructed a finitely presented group that contains coarsely an infinite expander graph but is not coarsely embedded into any Hilbert space. The construction involves random groups defined as quotients of hyperbolic groups by *randomly labeled expander graphs*. Details are given by G. Arzhantseva and T. Delzant in [AD08].

One of my research projects is to generalize an intermediate result in the construction, showing that there are *phase transitions* in the expander graph random group model.

The expander graph random groups. Let $X = \{x_1, \ldots, x_m\}$ be a set of $m \geq 2$ elements and F_m the free group generated by X. Consider the *randomly X-labeled expander graph* $\Gamma(p, q, j)$ defined as follows: $p, q \geq 3$ are distinct prime numbers and $j \geq 1$ is an integer. Let $C(p, q)$ be the Cayley graph of the projective general linear group $PGL_2(q)$ over the field of q elements, for a particular set of $(p + 1)$ generators. It is a $(p + 1)$-regular expander graph on $q^2(q - 1)$ vertices with girth (minimal simple cycle length) $\rho(p, q) \sim 4 \log_p q$ (c.f. [LPS88], [Val97]). The graph $\Gamma(p, q, j)$ is obtained from $C(p, q)$ by dividing every edge into j small edges, and by randomly labeling every small (oriented) edge by $X^{\pm}$. An *expander graph random group* is defined by the quotient

$G_q(m,p,j) := F_m/\Gamma(p,q,j)$, and we are interested in the asymptotic behaviors when the number q goes to infinity.

When the number of divisions j is very large, an expander graph random group $G_q(m,p,j)$ looks like a low-density random group. In particular, such a group satisfies some *small cancellation condition* and is *hyperbolic*.

Theorem 13 (M. Gromov, [Gro03])**.** *For any integer $m \geq 2$ and any prime number $p \geq 3$:*

(a) For any $0 < \lambda < 1$, there exists j_λ such that, with high probability,for any $j \geq j_\lambda$, the random group $G_q(m,p,j)$ satisfies the $C'(\lambda)$ graphical small cancellation condition (c.f. [AD08, Definition 2.3]).

(b) There exists j_0 large enough such that, with high probability, the random group $G_q(m,p,j)$ is non-elementary hyperbolic.

Are there phase transitions? There are three parameters in an expander graph random group: the number of generators m, the integer p that decides the valency of the graph ($(p+1)$-regular), and the division number j. The aim is to find out an analog of *density* in the expander graph random groups, and to exhibit phase transition phenomena in these groups.

Question 14. *Are there phase transitions for the two properties of Theorem 13? More precisely, can we find a critical j_λ such that if $j < j_\lambda$, then $G_q(m,p,j)$ is not graphical $C'(\lambda)$; and a critical j_0 such that if $j < j_0$, then $G_q(m,p,j)$ is not hyperbolic, or even trivial?*

The same question can be proposed for any phase transition that appeared in the density model: free subgroup problems, the existence of van Kampen diagrams, etc.

Here are some speculations on the question. Let us start with the $C'(\lambda)$ graphical small cancellation condition.

Let $\lambda < 1/2$. As the graph $C(p,q)$ is $(p+1)$-regular with $q^3 = p^{\frac{3}{4j}\rho_q + O(1)}$ vertices and he girth of $\Gamma(p,q,j)$ is $\rho_q = 4j\log_p q + O(1)$ (c.f. [LPS88], [Val97]), the number of simple paths of length $\lambda\rho_q$ on $\Gamma(p,q,j)$ is $C_q = p^{\frac{1}{j}\left(1+\frac{3}{4\lambda}\right)\lambda\rho_q + O(1)}$.

The set E_q of non-reduced words of $X_m^\pm$ of length $\lambda\rho_q$ is with cardinality $(2m)^{\lambda\rho_q}$. Let A_q be the set of words obtained by reading the paths of C_q. It is a *random subset* of E_q.

Guess 15. *The self-intersection of A_q does not affect its density. That is to say, the density of A_q is given by the cardinality of C_q, which gives*

$$\operatorname{dens}(A_q) = \frac{1}{j}\left(1 + \frac{3}{4\lambda}\right)\log_{2m} p.$$

Let B_q be the set of words of length $\lambda\rho_q$ that are equal to the identity in F_m. It is a fixed subset of E_q with

$$\operatorname{dens}(B_q) = \log_{2m}(2\sqrt{2m-1}).$$

Studying the $C'(\lambda)$ graphical small cancellation condition is to ask if A_q has self-intersection after reduction. For "regular enough" random subsets, it happens at the same density that A_q intersects B_q.

Guess 16. *The intersection between the random subset A_q and the fixed subset B_q satisfies Gromov's intersection formula, and A_q has self-intersection after reduction if and only A_q intersects B_q.*

If the above two guesses are true, then the $C'(\lambda)$ condition has a "phase transition", determined by the critical value $\operatorname{dens}(A_q) + \operatorname{dens}(B_q)$ that equals to

$$c(m,p,j,\lambda) = \frac{1}{j}\left(1 + \frac{3}{4\lambda}\right)\log_{2m} p + \log_{2m}(2\sqrt{2m-1}).$$

Guess 17. *If $c(m, p, j, \lambda) < 1$, then $G_q(m, p, j)$ satisfies $G'(\lambda)$; while if $c(m, p, j, \lambda) > 1$, then $G_q(m, p, j)$ does not satisfy $C'(\lambda)$.*

The triviality-hyperbolicity phase transition is more complicated. For the hyperbolicity, we can not rely on the small cancellation theory (that $C'(1/6)$ implies hyperbolicity). Establishing an isoperimetric inequality as in [Gro93, §9.B] and [Oll04, §2] seems to be the solution, but the study of abstract van Kampen diagrams with non-reduced relators is still not clear.

Project 18. *Establish an isoperimetric inequality for random group presentations with non-reduced relators, in the density model.*

Let R_q be the set of words read on simple cycles of $\Gamma(p, q, j)$ of length between ρ_q and the diameter of the graph. By estimating the first Betti number of the graph, we have

$$\mathrm{dens}(R_q) = \frac{7}{4j} \log_{2m} p.$$

To show the triviality, we want to find two words in R_q that differ by one letter after reduction. And again it happens at the same density that R_q intersects B_q, the set of words of length ρ_q that equal to the identity in F_m, for "regular enough" random subsets. So this time the guess is that the critical value is

$$c(m, p, j) = \frac{7}{4j} \log_{2m} p + \log_{2m}(2\sqrt{2m-1}).$$

Guess 19. *If $c(m, p, j) < 1$, then $G_q(m, p, j)$ is non-elementary hyperbolic; while if $c(m, p, j) > 1$, then $G_q(m, p, j)$ is trivial.*

Remark that $c(m, p, j)$ is the critical value for the $C'(\lambda)$ condition with "$\lambda = 1$", which coincides with the situation in the Gromov density model.

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