Phase transitions in random groups: free subgroups and van Kampen 2-complexes

Tsung-Hsuan Tsai

Institut de Recherche Mathématique Avancée
Université de Strasbourg

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What is a random group?

Definition

A random group G is a **random variable** *with values in a given (finite) set of groups.*

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A random group G is a **random variable** *with values in a given (finite) set of groups.*

For a group property P, we are interested in

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A random group is often constructed by a presentation with fixed generators and random relators:

$$G = \langle \underbrace{x_1, \ldots, x_m}_{\text{fixed}} \mid \underbrace{r_1 \ldots, r_k}_{\text{random}} \rangle.$$

Relators considered are cyclically reduced.

Asymptotic behaviors

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Definition

Let $(G_\ell)_{\ell \geq 1}$ be a sequence of random groups defined by

$$G_\ell = \langle x_1, \ldots, x_m | R_\ell \rangle$$

where R_ℓ is a random set of cyclically reduced relators of lengths at most ℓ.

Let $(P_\ell)_{\ell \geq 1}$ be a sequence of group properties. We say that G_ℓ satisfies P_ℓ **asymptotically almost surely** *(**a.a.s.**) if*

$$\mathbf{Pr}(G_\ell \text{ satisfies } P_\ell) \xrightarrow[\ell \to \infty]{} 1.$$

The density model of random groups

Definition (Gromov 1993)

Let $m \geq 2$, $d \in [0,1]$. A sequence of random groups $(G_\ell(m,d))$ with m generators **at density** d is defined by

$$G_\ell(m,d) = \langle x_1, \ldots, x_m | R_\ell \rangle$$

where R_ℓ is a random set of cyclically reduced relators of lengths at most ℓ, with

$$|R_\ell| = \lfloor (2m-1)^{d\ell} \rfloor,$$

uniformly chosen among all possible choices.

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uniformly chosen among all possible choices.

Denote B_ℓ as the set of cyclically reduced words of length *at most* ℓ, we have $|B_\ell| = (2m-1)^{\ell + O(1)}$, so $|R_\ell| = |B_\ell|^{d+o(1)}$.

The first result: phase transition at density 1/2

Theorem (Gromov 1993)

- *If* $d > 1/2$, *then a.a.s.* $G_\ell(m,d)$ *is a trivial group.*
- *If* $d < 1/2$, *then a.a.s.* $G_\ell(m,d)$ *is non-elementary hyperbolic and torsion-free. In addition, its presentation is aspherical.*

More precisely (Ollivier 2007), a.a.s. the Cayley graph of $G_\ell(m,d)$ is δ-hyperbolic with

$$\delta = \frac{4\ell}{1-2d}.$$

The first result: phase transition at density $1/2$

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Question (Gromov 2003)

Is there any other interesting phase transition?
For a sequence of group properties (P_ℓ), does there exist a critical density d_c such that

- *If $d < d_c$, then a.a.s. $G_\ell(m,d)$ satisfies P_ℓ;*
- *If $d > d_c$, then a.a.s. $G_\ell(m,d)$ does not satisfy P_ℓ?*

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If B is a finite dimensional vector space over a finite field and X is a affine subspace, then

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If R, R' are affine subspaces in *general position*, then

$$\dim(R \cap R') = \dim R + \dim R' - \dim B,$$

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The intersection formula

Metatheorem (The intersection formula, Gromov 1993)

Independent "random subsets" R and R' in a finite set B satisfy

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Let R, R' be uniform distributions of cardinality $|B|^d$, $|B|^{d'}$, then $\mathbf{Pr}(R \cap R' = \emptyset) > 0$.

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Let R, R' be uniform distributions of cardinality $|B|^d$, $|B|^{d'}$, then $\mathbf{Pr}(R \cap R' = \emptyset) > 0$. We have

$$Pr\Big(\big|\operatorname{dens}(R \cap R') - (d + d' - 1)\big| \leq \varepsilon\Big) \xrightarrow[|B| \to \infty]{} 1$$

Sequences of random subsets

Let $\boldsymbol{B} = (B_\ell)$ be a sequence of finite sets with $|B_\ell| \to \infty$.
Let $\boldsymbol{R} = (R_\ell)$ be a sequence of random subsets of $\boldsymbol{B} = (B_\ell)$.

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Definition (Densable sequences of random subsets)

We say that $\boldsymbol{R} = (R_\ell)$ *is* **densable** *with density* d *if the sequence of random variables*

$$\operatorname{dens}_{B_\ell}(R_\ell) = \log_{|B_\ell|}(|R_\ell|)$$

converges in probability to the constant $d \in \{-\infty\} \cup [0, 1]$.
We denote

$$\operatorname{dens}_{\boldsymbol{B}} \boldsymbol{R} = d.$$

Examples of densable sequences

Each of the following sequences of random subsets $\boldsymbol{R} = (R_\ell)$ is *densable* of density d.

- (The uniform model) R_ℓ follows the uniform distribution in the set of subsets of B_ℓ of cardinality $\lfloor |B_\ell|^d \rfloor$.
- (The Bernoulli model) The events $\{r \in R_\ell\}$ through $r \in B_\ell$ are independent of the same probability $|B_\ell|^{d-1}$. (Note that $\mathbb{E}(|R_\ell|) = |B_\ell|^d$.)
- (The random map model) Let A_ℓ be a set with $|A_\ell| = \lfloor |B_\ell|^d \rfloor$. R_ℓ is the image of a random map from A_ℓ to B_ℓ, uniformly chosen among all possible maps.

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$\boldsymbol{R}$ is called **permutation invariant** if R_ℓ is measure invariant under the permutations of B_ℓ.

Formal statement of the intersection formula

Theorem (The intersection formula)

Let $\boldsymbol{R} = (R_\ell)$, $\boldsymbol{R}' = (R'_\ell)$ *be independent* **densable** *sequences of* **permutation invariant** *random subsets of the sequence of sets* $\boldsymbol{B} = (B_\ell)$.

If $\operatorname{dens} \boldsymbol{R} + \operatorname{dens} \boldsymbol{R}' \neq 1$, *then the sequence* $\boldsymbol{R} \cap \boldsymbol{R}'$ *is densable and permutation invariant.*

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- *If* $\operatorname{dens} \boldsymbol{R} + \operatorname{dens} \boldsymbol{R}' < 1$, *then a.a.s.* $R_\ell \cap R'_\ell = \emptyset$, *i.e.*

$$\operatorname{dens}(\boldsymbol{R} \cap \boldsymbol{R}') = -\infty.$$

- *If* $\operatorname{dens} \boldsymbol{R} + \operatorname{dens} \boldsymbol{R}' > 1$, *then a.a.s.* $R_\ell \cap R'_\ell \neq \emptyset$ *and*

$$\operatorname{dens}(\boldsymbol{R} \cap \boldsymbol{R}') = \operatorname{dens} \boldsymbol{R} + \operatorname{dens} \boldsymbol{R}' - 1.$$

The random-fixed intersection formula

Theorem (The random-fixed intersection formula)

Let $\boldsymbol{R} = (R_\ell)$ *be a densable sequence of* **permutation invariant** *random subsets. Let* $\boldsymbol{X} = (X_\ell)$ *be a densable sequence of* **fixed** *subsets.*

If $\operatorname{dens} \boldsymbol{R} + \operatorname{dens} \boldsymbol{X} \neq 1$, *then the sequence* $\boldsymbol{R} \cap \boldsymbol{X}$ *is densable.*

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Application to random groups

$B_\ell =$ the set of cyclically reduced relators of length at most ℓ, $|B_\ell| = (2m-1)^{d\ell + O(1)}$.

Definition (The permutation invariant density model of random groups)

Let $m \geq 2$, $d \in [0,1]$. A sequence of random groups $(G_\ell(m,d))$ with m generators of **density d** is defined by

$$G_\ell(m,d) = \langle x_1, \ldots, x_m | R_\ell \rangle$$

where $\boldsymbol{R} = (R_\ell)$ is a *densable* sequence of *permutation invariant* random subsets of $\boldsymbol{B} = (B_\ell)$ of density d.

Application to random groups

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Example: Let X_ℓ be the set of relators in B_ℓ of type $r = w^2$.
We have $|X_\ell| = (2m-1)^{d\ell/2+O(1)}$, so dens $\boldsymbol{X} = 1/2$.

- If $d + 1/2 < 1$, then a.a.s. there is no relator of type w^2 in R_ℓ.
- If $d + 1/2 > 1$, then a.a.s. there exists relators of type w^2 in R_ℓ.

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Magnus' Freiheitssatz

Theorem (The Freiheitssatz ("freedom theorem" in German), Magnus 1933)

Let G be a group defined by a presentation with a single cyclically reduced relator

$$G = \langle x_1, \ldots, x_m | r \rangle.$$

If x_m appears in r, then $x_1, \ldots, x_{m-1}$ freely generate a free subgroup of G.

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If x_m appears in r, then $x_1, \ldots, x_{m-1}$ freely generate a free subgroup of G.

Given a group presentation with m generators

$$G = \langle x_1, \ldots, x_m | R \rangle.$$

If the first $(m-1)$ generators $x_1, \ldots, x_{m-1}$ freely generate a free subgroup, then it is called a "Freiheitssatz presentation".

For few relator random groups

For few relator random groups (a particular case of density 0), there is a much stronger result:

Theorem (Arzhantseva-Ol'shanskii 1996)

Let (G_ℓ) be a sequence of random groups with m generators and k relators (k is fixed, independent of ℓ), defined by

$$G_\ell = \langle x_1, \ldots, x_m | r_1, \ldots, r_k \rangle$$

with $|r_i| \leq \ell$ randomly chosen (uniformly among all possible choices).

If $m \geq 2$ and $k \geq 1$, then a.a.s. **every** *$(m-1)$-generated subgroup of G_ℓ is free.*

In particular, a.a.s. the presentation of G_ℓ is a Freiheitssatz presentation.

For the density model of random groups

For the density model, copying the proof of Arzhantseva and Ol'shanskii:

Theorem

Let $(G_\ell(m,d))$ be a sequence of random groups at density d, defined by

$$G_\ell(m,d) = \langle x_1, \ldots, x_m | R_\ell \rangle.$$

If $d < \frac{1}{120m^2 \ln(2m)} \sim \frac{1}{m^2 \ln m}$, then a.a.s. every $(m-1)$-generated subgroup of G_ℓ is free.

In particular, if $d < \frac{1}{120m^2 \ln(2m)} \sim \frac{1}{m^2 \ln m}$, then a.a.s. the presentation of $G_\ell(m,d)$ is a Freiheitssatz presentation.

Question

Is the density $d \sim \frac{1}{m^2 \ln m}$ optimal for this property?

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More generally, $(m-1)$ can be replaced by any integer $1 \leq r \leq m-1$.

Question

Let $1 \leq r \leq m-1$. Does there exist a critical density $d(m,r)$ such that

- *if $d < d(m,r)$ then a.a.s. every r-generated subgroup of $G_\ell(m,d)$ is free;*
- *if $d > d(m,r)$ then a.a.s. there exists a non-free r-generated subgroup in $G_\ell(m,d)$?*

The phase transition at density d_r

Let $1 \leq r \leq m-1$. There is a **phase transition** at density

$$d_r = \min\left\{\frac{1}{2}, 1 - \log_{2m-1}(2r-1)\right\}.$$

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Theorem

Let $(G_\ell(m,d)) = (\langle x_1, \ldots, x_m | R_\ell \rangle)$ be a sequence of random groups with m generators at density d.

- *If $d < d_r$, then a.a.s. every r-generated subgroup of $G_\ell(m,d)$ with "small" generators (i.e. $H = \langle y_1, \ldots, y_r \rangle$ with lengths $|y_i| \leq \frac{d_r - d}{30r}\ell$) is free and quasi-convex.*

  In particular, a.a.s. the first r generators $x_1, \ldots, x_r$ freely generate a free subgroup.

- *If $d > d_r$, then a.a.s. the first r generators $x_1, \ldots, x_r$ generate the whole group $G_\ell(m,d)$ (which is not free).*

Proof for the case $d > d_r$

- If $d_r = 1/2$, then a.a.s. $G_\ell(m, d)$ is trivial. Suppose that $d_r = 1 - \log_{2m-1}(2r - 1)$.

Proof for the case $d > d_r$

- If $d_r = 1/2$, then a.a.s. $G_\ell(m,d)$ is trivial. Suppose that $d_r = 1 - \log_{2m-1}(2r-1)$.

- Let $X_\ell = \{x_{r+1}w \mid w \text{ is a word of } x_1, \ldots, x_r\} \subset B_\ell$.
  $|X_\ell| = (2r-1)^{\ell+O(1)}$, so
  $$\mathrm{dens}_{\boldsymbol{B}}\, \boldsymbol{X} = \log_{2m-1}(2r-1)$$
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- By the intersection formula, $\mathrm{dens}\, \boldsymbol{R} + \mathrm{dens}\, \boldsymbol{X} > d_r + \log_{2m-1}(2r-1) = 1$, so a.a.s. $R_\ell \cap X_\ell$ is not empty.

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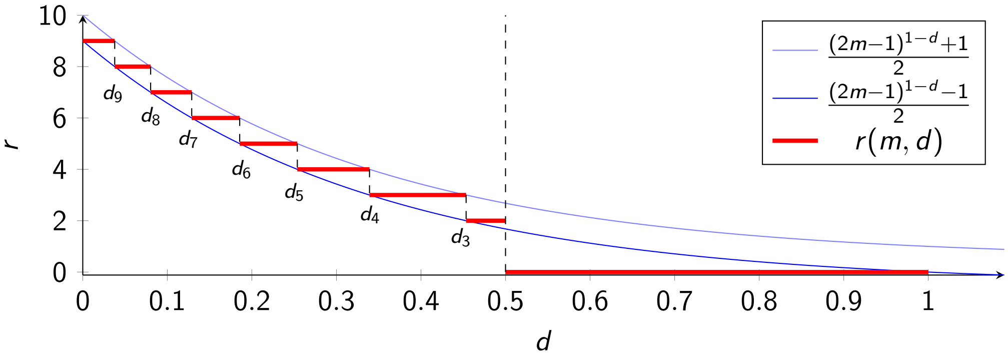
- Hence, a.a.s. x_{r+1} can be written as a word w^{-1} of $x_1, \ldots, x_r$ in $G_\ell(m, d)$.

- Apply the same argument to the other generators $x_{r+2}, \ldots, x_m$. □

The phase transition at density d_r

Let $r = r(m,d)$ be the maximal number such that a.a.s. $x_1, \ldots, x_r$ freely generate a free subgroup of $G_\ell(m,d)$.



$r(m,d)$ with $m = 10$ generators

Freiheitssatz for random groups

Corollary

If $d_r < d < d_{r-1}$, then a.a.s. the random group $G_\ell(m,d) = \langle x_1, \ldots, x_m | R_\ell \rangle$ has an aspherical presentation with r generators

$$\langle x_1, \ldots, x_r | R'_\ell \rangle$$

such that the first $r-1$ generators $x_1, \ldots, x_{r-1}$ freely generate a free subgroup.

That is to say, if d is not one of the d_r, then a.a.s. the random group $G_\ell(m,d)$ has a Freiheitssatz presentation.

Remark: In this presentation, the relator lengths in R'_ℓ vary from ℓ to ℓ^2.

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Remark: In this presentation, the relator lengths in R'_ℓ vary from ℓ to ℓ^2.

"[...] What does a random group look like? [...] Nothing like we have ever seen before."

—— M. Gromov, "Spaces and Questions" 2000.

Freiheitssatz for Random groups

In particular, if $d < d_{m-1} \sim \frac{1}{m \ln m}$, then a.a.s. the presentation defining $G_\ell(m,d)$ is a Freiheitssatz presentation.

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In particular, if $d < d_{m-1} \sim \frac{1}{m \ln m}$, then a.a.s. the presentation defining $G_\ell(m, d)$ is a Freiheitssatz presentation.

This bound is much larger than the previous bound (by the few relator model method) $\sim \frac{1}{m^2 \ln m}$.

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This bound is much larger than the previous bound (by the few relator model method) $\sim \frac{1}{m^2 \ln m}$.

Question

Is it true that, if $d < d_r$, then a.a.s. every r-generated subgroup of $G_\ell(m,d)$ is free?

The question is still open. If it is true, then at density $d_r < d < d_{r-1}$, a.a.s.

- every $(r-1)$-generated subgroup of $G_\ell(m,d)$ is free
- and $G_\ell(m,d)$ is r-generated by not free,

so a.a.s. the rank of $G_\ell(m,d)$ is r.

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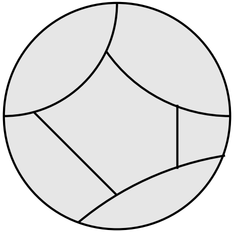
Van Kampen diagrams

Definition

A **van Kampen diagram** *with respect to a group presentation $G = \langle X | R \rangle$ is a finite, planar (embedded in $\mathbb{R}^2$) and simply connected 2-complex D such that:*

- *Every face has a preferred boundary loop (a starting point and an orientation).*
- *Every edge is labeled by a generator $x \in X^{\pm}$.*
- *Every face is labeled by a relator $r \in R$ such that the word read on the preferred boundary loop is r.*



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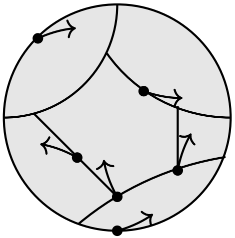
- *Every face has a preferred boundary loop (a starting point and an orientation).*
- *Every edge is labeled by a generator* $x \in X^{\pm}$.
- *Every face is labeled by a relator* $r \in R$ *such that the word read on the preferred boundary loop is* r.



Van Kampen diagrams

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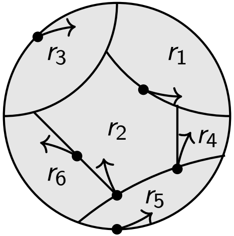
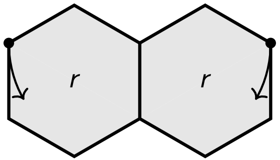


Reduced van Kampen diagrams

A pair of faces in a van Kampen diagram *reducible* if they have the same label and there is a common edge on their boundaries at the same position.



A van Kampen diagram is called *reduced* if there is no reducible pair of faces.

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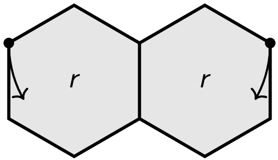

A van Kampen diagram is called *reduced* if there is no reducible pair of faces.

In the following, a 2-complex is *finite*, *planar* and *simply connected*.

Isoperimetric inequality

For a van Kampen diagram D,
$|D|$ = the number of faces,
$|\partial D|$ = the boundary length.

Proposition (Gromov 1993, Ollivier 2004)

Let $(G_\ell(m,d))$ be a sequence of random groups at density d. Let $K > 0$ be an integer.

If $d < 1/2$, then for any $s > 0$ a.a.s. every reduced van Kampen diagram D of $G_\ell(m,d)$ with $|D| \leq K$ satisfies the isoperimetric inequality

$$|\partial D| \geq (1 - 2d - s)|D|\ell.$$

The main question

Is the converse true?

Question

If a 2-complex D with $|D| \leq K$ (and $|\partial f| \leq \ell$ for any face f of D) satisfies the inequality

$$|\partial D| \geq (1 - 2d + s)|D|\ell.$$

does there exist (a.a.s.) a reduced van Kampen diagram of $G_\ell(m, d)$ whose underlying 2-complex is D?

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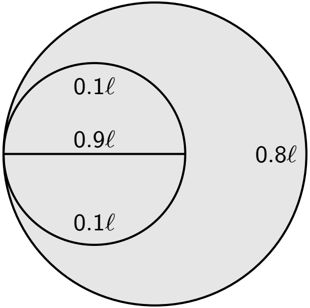
does there exist (a.a.s.) a reduced van Kampen diagram of $G_\ell(m, d)$ whose underlying 2-complex is D?

It is not true in general.

Counterexample

Let $(G_\ell(m, d))$ be a sequence of random groups at density $d = 0.4$.
Let D be the following 2-complex.

D



$$0.8\ell > (1 - 2 \times 0.4) \times 3\ell = 0.6\ell$$

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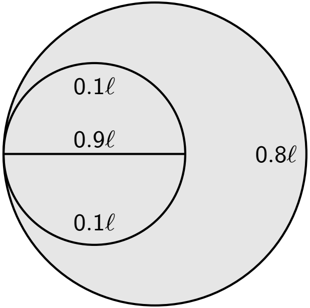
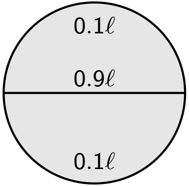


D'



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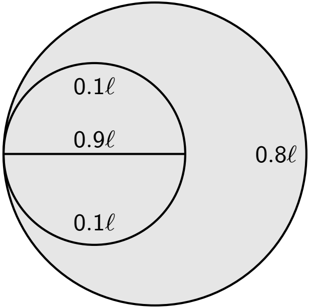

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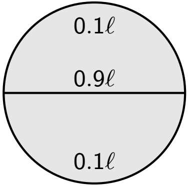

$$0.8\ell > (1 - 2 \times 0.4) \times 3\ell = 0.6\ell$$

$$0.2\ell < (1 - 2 \times 0.4) \times 2\ell = 0.4\ell$$

D' is not a v.K. diagram of $G_\ell(m, d)$, so D neither.

Geometrical form of 2-complexes

Consider a sequence of 2-complexes $\boldsymbol{D} = (D_\ell)$ of the same "geometrical form".

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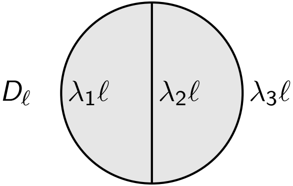
i.e.

- They have the same topological form.
- For any face f of D_ℓ, $|\partial f| \leq \ell$.
- The corresponding *maximal arc lengths* are $\lambda_1\ell, \ldots, \lambda_k\ell$.



(A maximal arc = a simple path passing by vertices of valency 2, with endpoints of valency $\neq 2$.)

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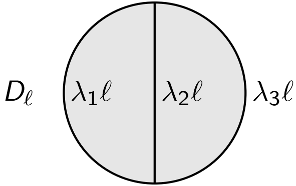

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Definition (Density of a sequence of diagrams)

The **density** *of* $\boldsymbol{D}$ *is*

$$\operatorname{dens} \boldsymbol{D} = \frac{\sum_{i=1}^k \lambda_i}{|\boldsymbol{D}|} \quad \left(= \lim_{\ell \to \infty} \frac{\operatorname{Edge}(D_\ell)}{|D_\ell| \ell} \right).$$

Critical density

According to the counterexample, every sub-2-complex should be considered.

Definition

The **critical density** *of* $\boldsymbol{D}$ *is*

$$\operatorname{dens}_c(\boldsymbol{D}) = \min_{\boldsymbol{D}' \leq \boldsymbol{D}} \operatorname{dens}(\boldsymbol{D}').$$

Note that $\operatorname{dens}_c \boldsymbol{D} \leq \operatorname{dens} \boldsymbol{D}$.

Phase transition for the existence of diagrams

Theorem

Let $(G_\ell(m,d))$ *be a sequence of random groups at density* d*. Let* $K > 0$*.*
Let $\boldsymbol{D} = (D_\ell)$ *be a sequence of 2-complexes of the same geometrical form, with* K *faces.*

- *If* $d > 1 - \mathrm{dens}_c(\boldsymbol{D})$*, then a.a.s. there exists a reduced van Kampen diagram of* $G_\ell(m,d)$ *whose underlying 2-complex is* D_ℓ*.*
- *If* $d < 1 - \mathrm{dens}_c(\boldsymbol{D})$*, then a.a.s. there is no reduced van Kampen diagram of* $G_\ell(m,d)$ *whose underlying 2-complex is* D_ℓ*.*

Phase transition for existence of van Kampen diagrams

Corollary (of the first point)

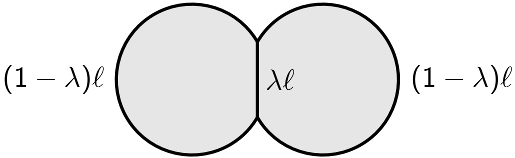
Let D be a 2-complex with K faces and $|\partial f| = \ell$ for every face f of D_ℓ. For any $s > 0$, if **every sub-2-complex** *D' of D satisfies the isoperimetric inequality*

$$|\partial D'| \geq (1 - 2d + s)|D'|\ell,$$

then a.a.s. in $G_\ell(m, d)$, there exists a reduced van Kampen diagram of whose underlying 2-complex is D.

Applications: Phase transition for $C'(\lambda)$ condition

The random group $G_\ell(m,d)$ *does not* satisfy $C'(\lambda)$ if there exists a reduced van Kampen of the following form D_ℓ.



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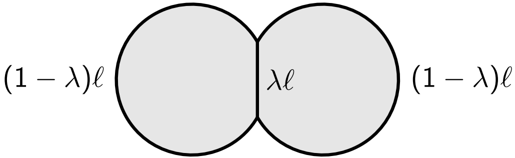

$\text{dens}_c\, \boldsymbol{D} = 1 - \lambda/2$

Application (Gromov 1993, Bassino-Nicaud-Weil 2020)

Let $0 < \lambda < 1$. There is a phase transition at density $d = \lambda/2$:

(i) *If $d < \lambda/2$, then a.a.s. $G_\ell(m, d)$ satisfies $C'(\lambda)$.*

(ii) *If $d > \lambda/2$, then a.a.s. $G_\ell(m, d)$ does not satisfy $C'(\lambda)$.*

Applications: Phase transition for $C(p)$ condition

Application

Let $p \geq 2$ be an integer. There is a phase transition at density $d = \frac{1}{p+1}$:

(i) *If $d < \frac{1}{p+1}$, then a.a.s. $G_\ell(m, d)$ satisfies $C(p)$.*

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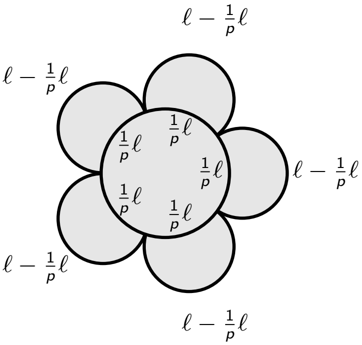

$$\mathrm{dens}_c\, \boldsymbol{D} = \frac{p}{p+1}$$

For non-planar 2-complexes

The phase transition for existence of diagrams holds for *non-planar* 2-complexes, with some more conditions...

For non-planar 2-complexes

The phase transition for existence of diagrams holds for *non-planar* 2-complexes, with some more conditions...

The condition $|D| \leq K$ is not enough and should be replaced by:

Definition (Complexity)

A 2-complex D is of complexity K if:

- $|D| \leq K$.
- *The number of maximal arcs of D is bounded by K.*
- *For any face f of D, the boundary path ∂f is divided into at most K maximal arcs.*

Remark: If D is planar and simply connected with $|D| \leq K$, then it is with complexity $6K$.

Phase transition for existence of van Kampen 2-complexes

Theorem

Let $(G_\ell(m,d))$ *be a sequence of random groups at density* d*. Let* $K > 0$*.*
Let $\boldsymbol{D} = (D_\ell)$ *be a sequence of 2-complexes of the same geometrical form, of complexity* K*.*

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Let $\boldsymbol{D} = (D_\ell)$ be a sequence of (non-planar) 2-complexes of the same geometrical form, with complexity K.
If D_ℓ does *not* collapse to a graph, then there exists a sub 2-complex such that every edge is adjacent to at least 2 faces. So $\text{dens}_c\, \boldsymbol{D} \leq \frac{1}{2} < 1 - d$.

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Proposition

If D_ℓ is the underlying 2-complex of a reduced van Kampen 2-complex of $G_\ell(m, d)$ with $d < 1/2$, then D_ℓ collapses to a graph.

Table of Contents

Existence of surfaces with a fixed genus

Let $\boldsymbol{S} = (S_\ell)$ be a sequence of 2-complexes of the same geometrical form. If S_ℓ is a surface of genus g with $|S_\ell| \leq K$, then it is with complexity $10gK$, and we have

$$\operatorname{dens}_c \boldsymbol{S} \leq \frac{1}{2}.$$

Proposition

Let $(G_\ell(m,d))$ be a sequence of random groups at density d. Let $g \geq 0$, $K \geq 1$.

Let $g \geq 0$. If $d < 1/2$, then a.a.s. $G_\ell(m,d)$ does not contain any surface S_ℓ of genus g with $|S_\ell| \leq K$.

Existence of surfaces with a fixed genus

Can we remove the condition $|S_\ell| \leq K$?

Question

Given $g \geq 0$. Is it true that, if $d < 1/2$, then a.a.s. $G_\ell(m, d)$ does not contain any surface subgroup of genus g?

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It is true for $g = 0$, because a.a.s. the presentation of $G_\ell(m, d)$ is aspherical.
It is true for $g = 1$, because a.a.s. the $G_\ell(m, d)$ is hyperbolic (contain no $\mathbb{Z}^2$).

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The genus g should be given in advance (i.e. should not depend on ℓ).

Theorem (Calegari-Walker 2015)

If $d < 1/2$, then a.a.s. $G_\ell(m,d)$ contains surface subgroups of genus $g = O(\ell)$.

The graph model of random groups

Let F_m be a free group with a generating set $X_m = \{x_1, \ldots, x_m\}$.

Let Γ be a graph labeled by $X_m^{\pm}$, then

$$F_m/\Gamma := F_m/\langle\langle \text{words of loops of } \Gamma \rangle\rangle$$

defines a group.

If Γ is a **randomly labeled graph**, then F_m/Γ is a random group.

The graph model of random groups

Ramanujan graph:

Let $p, q \geq 3$ be prime numbers congruent to 1 modulo 4, $p < q$, with the Legendre symbol $\left(\frac{p}{q}\right) = -1$.

$C(p, q) =$ the Cayley graph of $PGL_2(\mathbb{F}_q)$ with a certain set of $(p+1)/2$ generators.
$C(p, q, j) =$ divide every edge of $C(p, q)$ into j edges.
$\Gamma(p, q, j) =$ randomly (non-reduced) label of $C(p, q, j)$ by $X_m^{\pm}$.

$$G_q(m, p, j) = F_m / \Gamma(p, q, j).$$

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$$G_q(m, p, j) = F_m / \Gamma(p, q, j).$$

We are interested in the asymptotic behaviors of the sequence of random groups

$$(G_q(m, p, j))_{q \text{ prime}, q \neq p}.$$

The graph model of random groups

Theorem (Gromov 2003)

For any $m \geq 2$ and any $p \geq 3$, there exists j_0 large enough such that, for any $j \geq j_0$, a.a.s. (when $q \to \infty$) the random group $G_q(m, p, j)$ is non-elementary hyperbolic.

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Question

Is there a phase transition for this property?
More precisely, does there exist a function $c = c(m,p,j)$ such that

(i) *if $c > 1$, then a.a.s. $G_q(m,p,j)$ is trivial,*
(ii) *if $c < 1$, then a.a.s. $G_q(m,p,j)$ is non-elementary hyperbolic?*

A guess on $c = c(m, p, j)$

Let ρ_q be the girth (smallest simple cycle length) of $\Gamma(p, q, j)$.
Let R_q be the set of words read on simple cycles of $\Gamma(p, q, j)$ of length ρ_q, then dens $\boldsymbol{R} = (R_q)$ is the density of relators.

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Let X_q be the (fixed) set of non-reduced words of length ρ_q that are trivial in F_m.

Question

Does the intersection formula hold between $\boldsymbol{R}$ *and* $\boldsymbol{X}$ *(in* $\boldsymbol{B} = (B_q)$*)?*
If it does, is it true that the phase transition happens at

$$\begin{aligned} c(m, p, j) &= \operatorname{dens} \boldsymbol{R} \qquad + \operatorname{dens} \boldsymbol{X} \\ &= \frac{7}{4j} \log_{2m}(p) + \log_{2m}(2\sqrt{2m-1}) = 1\,? \end{aligned}$$

(?) (known)

Parallel geodesics and injectivity radius

Definition

Two bi-infinite geodesics in a hyperbolic space are called parallel if they have the same pair of limit points at the boundary and they have no intersection.
A set of $k \geq 2$ bi-infinite geodesics are called parallel if they are pairwise parallel.

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Proposition (Gruber-Mackay 2018)

If $d < \frac{11-\sqrt{41}}{12}$, then a.a.s. there exists an integer $k = k(d)$ such that the number of parallel geodesics in a **random triangular group** *at density d is bounded by k.*
In particular, a.a.s. the injectivity radius (smallest stable length) is at least $1/k$.

Parallel geodesics and injectivity radius

In the Gromov density model:

Proposition

If $d < 1/4$, then a.a.s. the number of parallel geodesics in a random group at density d is bounded by

$$k = 2 + \frac{2d}{1-4d}.$$

In particular, a.a.s. the injectivity radius is at least $1/k$.

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Is there a phase transition?

Question

If $d > \frac{1}{4}$, is it true the number of parallel geodesics is not uniformly bounded (when $\ell \to \infty$) ? (i.e. for any $k \geq 0$ a.a.s. the number of parallel geodesics is larger than k.)