Introduction to Geometric Group Theory

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Abstract

Geometric group theory studies groups from a geometric perspective. Given a finitely generated infinite group (such as $\mathbb{Z}^n$, free groups, surface groups, etc.), one constructs a metric space on which the group acts "nicely", and from the properties of this space one extracts properties of the group.

In this course, we will introduce some basic notions of geometric group theory and discuss a number of important examples of finitely presented groups. By the end of the course, students should be able to visualize these groups as geometric objects and recognize them through their geometric properties.

We will assume only basic knowledge of group theory (quotients, isomorphism theorems, ...) and of topology on metric spaces (connectedness, compactness, quotient spaces, ...). Some familiarity with algebraic topology would be helpful but is not required.

There will not be proofs for **Propositions** and **Corollaries** in this lecture note. They are also exercises! For each section (except for Section 3), one **Theorem** will be attributed as Homework.

We will roughly cover the textbook [Löh] *Geometric Group Theory: An Introduction* by Clara Löh, Chapters 1 to 7. Below are some useful references:

- **Undergraduate**
  - [Clay–Margalit] *Office Hours with a Geometric Group Theorists*
  - [Armstrong] *Groups and Symmetry*
- **Algebraic Topology**
  - [Massey] *A Basic Course in Algebraic Topology*
  - [Hatcher] *Algebraic Topology*
- **Graduate**
  - [Bridson–Haefliger] *Metric Spaces of Non-Positive Curvature*
  - [Druţu–Kapovich] *Geometric Group Theory*
  - [de la Harpe] *Topics in Geometric Group Theory*
  - [Lyndon–Schupp] *Combinatorial Group Theory*
  - [Serre] *Trees*
  - [Ol'shanskii] *Geometry of Defining Relations in Groups*
  - [ed. Ghys–Haefliger–Verjovsky] *Group Theory from a Geometrical Viewpoint*
- **French**
  - [Coornaert–Delzant–Papadopoulos] *Géométrie et Théorie des Groupes: Les Groupes Hyperboliques de Gromov*
  - [Ghys–de la Harpe] *Sur les Groupes Hyperboliques d'après Mikhael Gromov*
- **Gromov**
  - [Gromov 1987] *Hyperbolic Groups*, in *Essays in Group Theory*
  - [Gromov 1993] *Asymptotic Invariants of Infinite Groups*

Contents

1 Basics

1.1 Action!

1.1.1 Group actions on sets

A group action can be thought of as the "motion" of a space by a group.

Definition 1.1. *An **action** of a group G on a set X, denoted by $G \curvearrowright X$, is a function*

$$\alpha \colon G \times X \to X$$

where $\alpha(g, x)$ is written as $g \cdot x$ or gx, such that for all $g, h \in G$ and all $x \in X$,

- $1_G \cdot x = x$,
- $g \cdot (h \cdot x) = (gh) \cdot x$.

Equivalently, an action is a group homomorphism $\rho \colon G \to \mathrm{Sym}(X)$ where $\mathrm{Sym}(X)$ denotes the group of bijections of X, called the *symmetric group* of X; and $\rho(g)(x) = g \cdot x$.

Remark. What we have defined is a *left* action. We can also define a *right* action, denoted by $X \curvearrowleft G$, as a function $\alpha \colon X \times G \to X$ satisfying $x \cdot 1_G = x$ and $(x \cdot g) \cdot h = x \cdot (gh)$. A right action is then equivalent to an *anti*-homomorphism $\rho \colon G \to \mathrm{Sym}(X)$. That is, $\rho(gh) = \rho(h) \circ \rho(g)$.

Example. Some examples of group actions.

- $\mathrm{Sym}(X) \curvearrowright X$.
- $\mathrm{Sym}(X) \curvearrowright \mathcal{P}(X)$ where $\mathcal{P}(X)$ is the set of all subsets of X.
- $G \curvearrowright G$ by left multiplication, $G \curvearrowleft G$ by right multiplication.
- $\mathrm{Aut}(G) \curvearrowright G$.
- $\mathbb{Z} \curvearrowright \mathbb{R}$ by translation, $\mathbb{Z}^n \curvearrowright \mathbb{R}^n$ by translation.
- $\mathrm{Homeo}(\mathbb{R}) \curvearrowright \mathbb{R}$.
- $\mathbb{Z} \curvearrowright \mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ by rotation: $n \cdot z = e^{ni\theta} z$ where $\theta \in \mathbb{R}$.
- $\mathrm{GL}_n(\mathbb{R}) \curvearrowright \mathbb{R}^n$ by matrix multiplication.

1.1.2 Orbits, stabilizers, fixed points

Definition 1.2. *Given an action $G \curvearrowright X$.*

- *The **orbit** of $x \in X$ by G is the set*

$$\mathrm{Orb}_G(x) = G \cdot x := \{g \cdot x \mid g \in G\} \subset X.$$

- *The **stabilizer** of $x \in X$ by G is the set*

$$\mathrm{Stab}_G(x) := \{g \in G \mid gx = x\} \subset G.$$

Proposition. Stabilizers are subgroups.

Exercise. Show that $\mathrm{Stab}(gx) = g\,\mathrm{Stab}(x)g^{-1}$.
Exercise. Find orbits and stabilizers of $\mathrm{Sym}(X) \curvearrowright \mathcal{P}(X)$.

Definition 1.3. *Given $G \curvearrowright X$.*

- *The set of* ***fixed points*** *of $g \in G$ is the set*

$$\mathrm{Fix}(g) := \{x \in X \mid gx = x\} \subset X.$$

- *A* ***global fixed point*** *is an element $x \in X$ fixed by all $g \in G$. The set of global fixed points is then*

$$\bigcap_{g \in G} \mathrm{Fix}(g) = \{x \in X \mid \forall g \in G, gx = x\} \subset X.$$

Exercise. Find fixed points and global fixed points of $\mathrm{Sym}(X) \curvearrowright \mathcal{P}(X)$.

Theorem 1.4. *Let $G \curvearrowright X$ and $x \in X$. Denote by $G/\mathrm{Stab}(x)$ the set of left cosets of $\mathrm{Stab}(x)$ in G. Then the map*

$$\begin{aligned} G/\mathrm{Stab}(x) &\to \mathrm{Orb}(x) \\ g\,\mathrm{Stab}(x) &\mapsto gx \end{aligned}$$

is well-defined and bijective.

Proof. There are three things to be checked in exercises of this kind: well-definedness, injectivity, and surjectivity. In many cases one also needs to check that the map is a homomorphism of some structure, but this is not the case here.

- **Well-defined.** Suppose $g\,\mathrm{Stab}(x) = g'\,\mathrm{Stab}(x)$. Then $g^{-1}g' \in \mathrm{Stab}(x)$, hence $(g^{-1}g')x = x$. Multiplying by g on the left gives $g'x = gx$. Therefore, the image of a coset does not depend on the chosen representative.
- **Surjectivity.** Let $y \in \mathrm{Orb}(x)$. By definition of the orbit, there exists $g \in G$ such that $y = gx$. Thus y is the image of the coset $g\,\mathrm{Stab}(x)$.
- **Injectivity.** Suppose that $g\,\mathrm{Stab}(x)$ and $g'\,\mathrm{Stab}(x)$ have the same image, i.e. $gx = g'x$. Then $g^{-1}g'x = x$, so $g^{-1}g' \in \mathrm{Stab}(x)$, which implies $g\,\mathrm{Stab}(x) = g'\,\mathrm{Stab}(x)$.

□

Corollary 1.5 (Orbit-Stabilizer). *If G is a finite group, then for any $x \in X$,*

$$|G| = |\mathrm{Stab}(x)||\mathrm{Orb}(x)|.$$

Exercise (Cauchy's Theorem). Let G be a finite group and let p be a prime dividing $|G|$. Let

$$X = \{(g_1, \dots, g_p) \in G^p \mid g_1 g_2 \cdots g_p = e\}.$$

Define an action of $\mathbb{Z}/p\mathbb{Z}$ on X by

$$k \cdot (g_1, \dots, g_p) = (g_{k+1}, \dots, g_k).$$

- Show that every orbit of this action has size either 1 or p.
- Let $F \subset X$ be the set of global fixed points (i.e. of orbit size 1). Show that p divide $|F|$.
- Justify that $|F| = |\{g \in G \mid g^p = e\}|$. Conclude that there exists $g \in G$, $g \neq e$, such that $g^p = e$.

1.1.3 Free, transitive, faithful

Definition 1.6. *An action $G \curvearrowright X$ is said to be*

- ***free*** *if $gx \neq x$ for any $g \in G$ and $x \in X$;*
- ***transitive*** *if for any $x, y \in X$ there exists $g \in G$ such that $gx = y$;*
- ***faithful*** *if for any $g \in G$ there exists $x \in X$ such that $gx \neq x$.*

Exercise. Determine if $\mathrm{Sym}(X) \curvearrowright \mathcal{P}(X)$ is free, transitive, or faithful.

Proposition. Every free action is a faithful action.

Proposition. If $G \curvearrowright X$ freely and transitively, then there is a natural bijection from G to X.

Proposition. An action $G \curvearrowright X$ is faithful if and only if the corresponding homomorphism $\rho\colon G \to \mathrm{Sym}(X)$ is a monomorphism. In any case, the action $G/\ker(\rho) \curvearrowright X$ defined by $g\ker(\rho) \cdot x := g \cdot x$ is well defined and faithful.

Proposition. Let $G \curvearrowright X$ be an action.

- The action is free if and only if for every $g \in G\backslash\{1_G\}$, $\mathrm{Fix}(g) = \varnothing$.
- The action is transitive if and only if there exists $x \in X$ such that $\mathrm{Orb}(x) = X$; if and only if for all $x \in X$, $\mathrm{Orb}(x) = X$.
- The action is faithful if and only if $\bigcap_{x \in X} \mathrm{Stab}(x) = \{1_G\}$.

Theorem 1.7. *1. Every transitive action $G \curvearrowright X$ is "equivalent" to an action $G \curvearrowright G/H$ by left multiplication where H is a subgroup and G/H is the set of left cosets.*
2. $G \curvearrowright G/H$ and $G \curvearrowright G/K$ are "equivalent" if and only if H and K are conjugate in G.

Proof. Homework. □

Example (Dihedral group)**.**
The **dihedral group** D_n is the set of isometries of an n-gon, acting naturally on the n-gon. It consists of n rotations (including the identity) and n reflections. It's the subgroup of $\mathrm{Homeo}(\mathbb{S}^1)$ generated by $r\colon e^{i\theta} \mapsto e^{i\theta + i\frac{2\pi}{n}}$ and $s\colon e^{i\theta} \mapsto e^{-i\theta}$.

We can also define ∞-gon as the real line $\mathbb{R}$ where the points on $\mathbb{Z}$ are marked; and the **infinite dihedral group** D_∞ as the set of isometries of $\mathbb{R}$ that preserves $\mathbb{Z}$. It's the subgroup of $\mathrm{Homeo}(\mathbb{R})$ generated by $r\colon x \mapsto x + 1$ and $s\colon x \mapsto -x$.

Note that in either case, $srs = r^{-1}$.

Exercise. Let $n \in \{3, 4, \dots\} \cup \{\infty\}$. Consider the action of D_n on an n-gon, which consists of n vertices and n edges. Show that:

- D_n acts transitively on the set of vertices, but not freely.
- D_n acts freely on the set of pairs of vertices if n is odd and not freely if n is even, but never transitively.
- D_n acts transitively on the set of edge, but not free.
- D_n acts freely and transitively on the set of *oriented* edges.

1.2 Metric

1.2.1 Metric spaces

Definition 1.8. *A* ***metric space*** (X, d) *is a set* X *together with a distance function*

$$d\colon X \times X \to \mathbb{R}$$

such that for all $x, y, z \in X$*,*

Positive definite:	$d(x, y) \geq 0$ *and* $d(x, y) = 0$ *if and only if* $x = y$*.*
Symmetry:	$d(x, y) = d(y, x)$*.*
Triangle inequality:	$d(x, z) \leq d(x, y) + d(y, z)$*.*

Given $x \in X$ and $r > 0$, the open ball of radius r about x is the set

$$B(x, r) := \{y \in X \mid d(x, y) < r\},$$

and the closed ball

$$\overline{B}(x, r) := \{y \in X \mid d(x, y) \leq r\}.$$

Associated to the metric d one has the topology whose basis is the set of open balls $B(x, r)$. Note that in this topology, $\overline{B}(x, r)$ may be strictly larger than the closure of $B(x, r)$. The metric space is said to be **proper** if every closed ball $\overline{B}(x, r)$ is compact.

Given a metric space (X, d), a subset $Y \subset X$ is naturally a metric space $(Y, d_{|Y \times Y})$, called a **subspace**. A subspace Y is called **bounded** if there exists $B > 0$ such that $d(x, y) \leq B$ for all $x, y \in Y$.

Example. The set $\mathbb{R}^n$ with the usual Euclidean metric

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

Example. Any set X together with the discrete metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

Compare $\overline{B(x, 1)}$ and $\overline{B}(x, 1)$.

Example. Let $\ell^2(\mathbb{R})$ denote the space of square–summable real sequences

$$\ell^2(\mathbb{R}) = \left\{ x = (x_n)_{n \geq 1} \,\middle|\, \sum_{n=1}^{\infty} x_n^2 < \infty \right\},$$

with the metric

$$d(x, y) = \sqrt{\sum_{n=1}^{\infty} (x_n - y_n)^2}.$$

Then $(\ell^2(\mathbb{R}), d)$ is *not* proper: the closed unit ball $\overline{B}(0, 1)$ is not compact, since the sequence $(e_n)_{n \geq 1}$ where $e_n = (0, \ldots, 0, 1, 0, \ldots)$ has no convergent subsequence.

1.2.2 Isometries and isometric actions

We now introduce the notion of *isometries*, that is, maps which preserve the metric structure of a space. This will allow us to define and study *isometric actions* on metric spaces.

Definition 1.9. *Let $f : X \to X'$ be a function from one metric space (X, d) to another (X', d').*

- *We say that f is an **isometric embedding** if*

$$d'(f(x), f(y)) = d(x, y) \quad \textit{for all } x, y \in X.$$

- *In addition, if there exists another isometric embedding $g : X' \to X$ such that*

$$g \circ f = Id_X \text{ and } f \circ g = Id_{X'},$$

  *then we say that f is an **isometry**.*

- *The two metric spaces (X, d) and (X', d') are said to be isometric.*
- *The set of isometries if a metric space (X, d) is denoted by $\mathrm{Isom}(X)$.*

Proposition.
- An isometric embedding is injective and continuous.
- A surjective isometric embedding is an isometry.
- $\mathrm{Isom}(X)$ is a subgroup of $\mathrm{Homeo}(X)$, the set of homeomorphisms of X.

Example. Let $m \leq n$ be integers. Then the canonical inclusion $\mathbb{R}^m \hookrightarrow \mathbb{R}^n$ is an isometric embedding.

Example. The map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (y, x)$ is an isometry.

Definition 1.10. *An isometric action of a group G on a metric space (X, d) is a group homomorphism $\rho : G \to \mathrm{Isom}(X)$..*

We say that G acts on X *by isometries* or *isometrically*.

Exercise. Let $G \curvearrowright (X, d)$ by isometries. Show that $gB(x, r) = B(gx, r)$ for any $g \in G$, $x \in X$, and $r > 0$.

Definition 1.11. *Let $G \curvearrowright (X, d)$ by isometry. The action is said to be*

- ***proper** if for any $x \in X$ and any $r > 0$, the set $\{g \in G \mid gB(x, r) \cap B(x, r) \neq \varnothing\} \subset G$ is finite.*
- ***cobounded** if there exists $x \in X$ and $r > 0$ such that $G \cdot B(x, r) = X$.*
- ***cocompact** if there exists a compact set K such that $G \cdot K = X$.*

Example. $\mathbb{Z}^n \curvearrowright \mathbb{R}^n$ by translations is proper and cocompact.

Proposition. An action is proper if and only if for any $x \in X$ and $r > 0$, the set $\{g \in G \mid gx \in B(x, r)\}$ is finite.

Exercise. Find a free action that is not proper, and a proper action that is not free.

Proposition. A transitive action is cocompact, and a cocompact action is cobounded.

Proof. Recall that every singleton is compact, and every compact set is bounded. ☐

Proposition. If the metric space is proper, then a cobounded action is cocompact.

1.2.3 Geodesic metric spaces

Definition 1.12 (Geodesic metric space)**.** *Let (X, d) be a metric space. The set of real numbers $\mathbb{R}$ is endowed with the usual metric $d_{\mathbb{R}}(a, b) = |a - b|$.*

- *A **geodesic** of X is a map γ from an interval $I \subset \mathbb{R}$ to X which is an isometric embedding.*

  That is, $d(\gamma(a), \gamma(b)) = |a - b|$ for every $a, b \in I$.

  If $I = [a, b]$ is finite, we say that the points $\gamma(a), \gamma(b) \in X$ are joined by the geodesic γ.

- *Let $L > 0$. An L**-local geodesic** of X is a map γ from an interval $I \subset \mathbb{R}$ to X such that for every sub-interval $J \subset I$ of length at most L, the restriction $\gamma_{|J}$ is a geodesic.*

- *The space X is called a **geodesic metric space** if every pair of points can be joined by a geodesic.*

Example. A great circle on a sphere is a local geodesic but not a geodesic.

Example. $\mathbb{R}^n$ with the Euclidean metric is a geodesic metric space, $\mathbb{R}^n \backslash \{0\}$ is not.

1.3 Graphs and Cayley graphs

1.3.1 Graphs

For the moment we consider graphs as 1-dimensional simplicial complexes, not oriented, without loops or multiple edges. For graphs as combinatorial 1-dimensional cell complexes, see [Lyondon-Schupp] or [Hatcher]. For graphs in the sense of Serre, see [Serre].

Definition 1.13. *A graph is a pair $\Gamma = (V, E)$ of disjoint sets where E is a set of subsets of V that contain exactly two elements. That is,*

$$E \subset \{e \mid e \subset V, |e| = 2\}.$$

A graph is **finite** if V and E are both finite. The **degree** (or **valence**) of a vertex $v \in V$, denoted by $\deg(v)$, is the number of its appearance in the edges. A graph is **locally finite** if every vertex has finite degree.

Exercise. For a finite graph,

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Some vocabularies:

- A **subgraph** of $\Gamma = (V, E)$ is a graph $\Gamma' = (V', E')$ such that $V' \subset V$ and $E' \subset E$.

  We can define naturally *inclusion*, *union*, and *intersection* between subgraphs according to their sets of vertices and sets edges.

- A **path** of length n is a sequence of vertices $v_0, v_1, \ldots, v_n$ such that $\{v_i, v_{i+1}\} \in E$.

- A **cycle** of length n is a path $v_0, v_1, \ldots, v_{n-1}$ such that $\{v_{n-1}, v_0\} \in E$.

- A **simple** path (resp. cycle) is a path (resp. cycle) whose vertices are different.

- A graph is **connected** if every pair of vertices can be connected by a path.

- A **tree** is a connected graph with no cycles.

- A graph is d-**regular** for some integer $d \geq 1$ if every vertex has degree d.

Example. Some examples of graphs.

- A cycle of length n.
- A 3-regular tree.
- A binary tree (finite or infinite).
- A complete graph on n vertices.

Exercise. For a finite tree, $|V| = |E| + 1$.

Definition 1.14. *A **graph (homo)morphism** φ from $\Gamma = (V, E)$ to $\Gamma' = (V', E')$ is a pair of maps $\varphi_0 \colon V \to V'$ and $\varphi_1 \colon E \to E'$ that are compatible in the sense that $\varphi_1(\{v_1, v_2\}) = \{\varphi_0(v_1), \varphi_0(v_2)\}$ for every edge $\{v_1, v_2\} \in E$.*
*A **graph isomorphism** is a graph morphism in which φ_0 and φ_1 are bijective.*

We can define similarly

- *monomorphisms* (= injective morphisms),
- *epimorphisms* (= surjective morphisms),
- *endomorphisms* (= morphisms from Γ ot itself), and
- *automorphisms* (= bijective morphisms from Γ to itself).

Denote by $\mathrm{Aut}(\Gamma)$ the group of automorphisms of the graph Γ.

Exercise. Let C_n be a cycle with $n \geq 3$. What is $\mathrm{Aut}(C_n)$?

Exercise. Let T_2 be a 2-regular tree. What is $\mathrm{Aut}(T_2)$?

Definition 1.15. *An action of a group on a graph $G \curvearrowright \Gamma$ is a pair of actions $G \curvearrowright V$ and $G \curvearrowright E$ that are compatible with the graph structure. That is, for any element $g \in G$ and any edge $e = \{v_1, v_2\} \in E$,*

$$g \cdot \{v_1, v_2\} = \{g \cdot v_1, g \cdot v_2\}.$$

Equivalently, an action $G \curvearrowright \Gamma$ is a group homomorphism $\rho \colon G \to \mathrm{Aut}(\Gamma)$

1.3.2 Graphs as metric spaces

Let $\Gamma = (V, E)$ be a connected graph. The vertex set V carries a natural metric structure (V, d_V): For any pair of vertices $u, v \in V$, we define $d_V(u, v)$ as the length of a shortest path from u to v.

Exercise. Show that d_V is a metric on V.

A connected graph $\Gamma = (V, E)$ can be *realized* as a geodesic metric space in a natural way. We denote this realization by $(X(\Gamma), d)$, and it contains (V, d_V) as an isometrically embedded subset. Informally, each vertex is a point and each edge is an isometric copy of the interval $[0, 1]$. The distance between two points is defined as the length of the shortest "path" joining them in the graph.

We regard V and E as discrete spaces and consider the topological space $V \sqcup (E \times [0, 1])$. For each edge $e = \{u, v\} \in E$, we impose the identifications $(e, 0) \sim u$ and $(e, 1) \sim v$. These identifications generate an equivalence relation $\sim$ (that is, we take the smallest equivalence relation containing them; in particular, if $(e, 0) \sim u$ and $(e', 1) \sim u$, then $(e, 0) \sim (e', 1)$ by transitivity). We then define

$$X(\Gamma) := (V \sqcup E \times [0, 1]) \, / \sim .$$

Remark. Note that for each edge $e = \{u, v\}$ there are two possible choices of identifications: either $(e, 0) \sim u$ and $(e, 1) \sim v$, or $(e, 0) \sim v$ and $(e, 1) \sim u$. The resulting metric spaces are isometric, since the map $t \mapsto 1 - t$ is an isometry of $[0, 1]$ interchanging the endpoints.

Points in $X(\Gamma)$ are then denoted by $\overline{v}$ for some $v \in V$ or by $\overline{(e, t)}$ for some $(e, t) \in E \times [0, 1]$. Since the graph is connected, for any vertex $v \in V$ we have $\overline{v} = \overline{(e, t)}$ for some $e \in E$ and some $t \in \{0, 1\}$.

We now define a distance function on $X(\Gamma)$. Let $x_1 = \overline{(e_1, t_1)}$ and $x_2 = \overline{(e_2, t_2)}$ be arbitrary points in $X(\Gamma)$. If $e_1 = e_2$, we define $d(x_1, x_2) = |t_1 - t_2|$. Otherwise, let u_1, v_1, u_2, v_2 be the endpoints of e_1 and e_2 so that

$$u_1 \sim (e_1, 0), \quad v_1 \sim (e_1, 1) \quad \text{and} \quad u_2 \sim (e_2, 0), \quad v_2 \sim (e_2, 1).$$

We define $d(x_1, x_2)$ to be the minimum of the following four quantities:

$$\begin{aligned} t_1 + d_V(u_1, u_2) + t_2, \\ t_1 + d_V(u_1, v_2) + (1 - t_2), \\ (1 - t_1) + d_V(v_1, u_2) + t_2, \\ (1 - t_1) + d_V(v_1, v_2) + (1 - t_2). \end{aligned}$$

That is, to go from x_1 to x_2, we first travel from x_1 to a vertex of e_1, then we travel through a path in the graph to arrive a vertex of e_2, then we go from this vertex to x_2.

Proposition. $(X(\Gamma), d)$ is a geodesic metric space. In addition,

- The inclusion $(V, d_V) \hookrightarrow (X(\Gamma), d)$ is an isometric embedding.
- For any edge e of Γ, the inclusion $\{e\} \times [0, 1] \hookrightarrow X(\Gamma)$ is an isometric embedding.

Proposition. Every graph morphism $\varphi : \Gamma \to \Gamma'$ induces a continuous map $\varphi : X(\Gamma) \to X(\gamma)$.

Exercise. Find an example of a graph monomorphism that is not an isometric embedding.

Proposition. Every graph automorphism of Γ induces an isometry of $X(\Gamma)$. That is, we have an inclusion $\mathrm{Aut}(\Gamma) \hookrightarrow \mathrm{Isom}(X(\Gamma))$.

Exercise. Find an example of an isometry of some $X(\Gamma)$ that is not induced by any automorphism of Γ.

1.3.3 Cayley graphs

A group G is **finitely generated** if there exists a finite subset $S \subset G$ such that every element $g \in G$ can be written as $g = s_1 s_2 \ldots s_n$ where $s_i \in S \cup S^{-1}$. For example, $\mathbb{Z}$ is finitely generated, but $\mathbb{Q}$ is not.

Definition 1.16. *Let G be a finitely generated group and S a finite generating set with $1_G \notin S$. The **Cayley graph** of G with respect to S, denoted by $\mathrm{Cay}(G, S)$, is the graph with*

- *G as the set of vertices, and*
- *there is an edge between g and h if and only if there exists $s \in S \cup S^{-1}$ such that $h = gs$.*

An edge (g, gs) is usually *labeled* (by s) and *oriented* (from g to gs) if $s \neq s^{-1}$. Different S gives different Cayley graphs.

The graph can be defined even if S is not finite, or if S is not a generating set. In this case:

Exercise. Show that $\mathrm{Cay}(G, S)$ is locally finite if and only if S is finite; is connected if and only if S is a generating set.

In the following, S is always a finite generating set of a group G if not otherwise stated.

Exercise. Draw the following Cayley graphs.

- $\mathrm{Cay}(\mathbb{Z}/n\mathbb{Z}, \{1\})$.
- $\mathrm{Cay}(D_n, \{r, s\})$.
- $\mathrm{Cay}(\mathbb{Z}, \{1\})$ and $\mathrm{Cay}(\mathbb{Z}, \{2, 3\})$.
- $\mathrm{Cay}(\mathbb{Z}^2, \{(1,0),(0,1)\})$ and $\mathrm{Cay}(\mathbb{Z}^2, \{(1,1),(2,1)\})$.
- $\mathrm{Cay}(\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}, \{(1,0),(0,1)\})$.

Remark. A group G with a finite generating set S (with $1_G \notin S$) has a natural metric. For any $g \in G$, define the **word length** of g with respect to S, denoted by $|g|_S$, as the minimal integer $n \geq 1$ such that g can be written as $g = s_1 \dots s_n$ for some $s_i \in S \cup S^{-1}$. Then the distance on G is defined by

$$d_S(g, h) := |g^{-1}h|_S.$$

Proposition. The map $(G, d_S) \to \mathrm{Cay}(G, S)$ which sends $g \in G$ to the vertex that represents g is an isometric embedding.

Proposition. Let G be a group with a (possibly infinite) generating set S. Consider the action $G \curvearrowright G$ by left multiplications.

1. The action extends naturally to $G \curvearrowright \mathrm{Cay}(G, S)$, as a graph action and hence as an isometric action.
2. The graph action $G \curvearrowright \mathrm{Cay}(G, S)$ is free and transitive on the vertices; and free on the edges if and only if S contains no elements of order 2.
3. The metric space $\mathrm{Cay}(G, S)$ is proper if and only if S is finite.
4. The isometric action $G \curvearrowright \mathrm{Cay}(G, S)$ is proper if and only if S is finite.
5. The action $G \curvearrowright \mathrm{Cay}(G, S)$ is always cobounded, hence cocompact if S is finite.

Theorem 1.17. *A group G is finitely generated if and only if it acts (by isometries) properly and coboundedly on a geodesic metric space (X, d).*

Proof. If a group is finitely generated, then it acts properly and coboundedly on its Cayley graph.

Conversely, suppose that a group G acts properly and coboundedly on a metric space X. By coboundedness, there exist $r > 0$ and a point $x \in X$ such that

$$G \cdot B(x, r) = X.$$

By properness, the set

$$S := \{ s \in G \mid d(x, sx) \leq 3r \}$$

is finite. We claim that S generates G.

Let $g \in G$. Subdivide a geodesic from x to gx into subsegments of length at most r. This yields a sequence of points $x_0, \dots, x_n$ in X such that $x_0 = x$, $x_n = gx$, and $d(x_i, x_{i+1}) \leq r$ for all $0 \leq i \leq n-1$. Since $G \cdot B(x, r) = X$, for each $1 \leq i \leq n-1$ there exists $g_i \in G$ with $x_i \in B(g_i x, r)$. Set $g_0 = 1_G$ and $g_n = g$.

For each i, we then have

$$d(g_i x, g_{i+1} x) \leq d(g_i x, x_i) + d(x_i, x_{i+1}) + d(x_{i+1}, g_{i+1} x) \leq 3r.$$

Hence $s_i := g_{i+1} g_i^{-1} \in S$. Consequently,

$$g = g_n g_{n-1}^{-1} g_{n-1} g_{n-2}^{-1} \cdots g_1 g_0^{-1} = s_{n-1} s_{n-2} \cdots s_0.$$

Therefore, G is generated by the finite set S. □

2 Free

Informally saying, the free group on a generating set S is the "freest possible" group generated by S, in the sense that no product can equal to the identity, except for the obvious ones like $gg^{-1} = 1$.

We will first define free groups in terms of reduced words with concatenation, and show that free groups can be characterized by universal properties.

2.1 Free groups

2.1.1 Monoids and words

Lets start by the definition of a *monoid*, which looks like a group but do not necessarily have inverses.

Definition 2.1. *A* ***monoid*** *is a set M with a binary operation $*\colon M \times M \to M$ which is associative and has an identity element ϵ, also called a* unit. *That is, for all $u, v, w \in M$,*

- $(u * v) * w = u * (v * w)$,
- $u * \epsilon = u = \epsilon * u$.

As usual, we can define *homomorphisms*, *monomorphisms*, *epimorphisms*, *isomorphisms*, *endomorphisms* and *automorphisms* of monoids.

Example. Some monoids.

- Every group is a monoid.
- $(\mathbb{N}, +)$, $(\mathbb{N}, \times)$, $(\mathbb{Z}, \times)$ are monoids but not groups.
- The set of endomorphisms from a set X to itself, with composition of functions.

Exercise. Show that the identity element is unique.

Let A be a non-empty set (of letters). The set of **words** on A is the set $W(A)$ of finite sequences of elements of A. The **length** of u is the length of the sequence, denoted by $|u|$. An element $u \in W(A)$ is written as a product $u = a_1 \dots a_n$ where $a_i \in A$. A *subword* is a consecutive subsequence $a_i \dots a_j$ where $1 \leq i \leq j \leq n$. For example, if $A = \{a, b, c\}$, then $abcc$ is a word of length 4, and bc is a subword of length 2.

By convention, the *empty word* ϵ is the only word of length 0; the word $aa \dots a$ (n times) may be written as a^n for some integer $n \geq 0$ where $a^0 = \epsilon$. The **concatenation** of two words $u = a_1 \dots a_n$ and $v = b_1 \dots b_m$ is the binary operation $*\colon W(A) \times W(A) \to W(A)$ defined by $u * v = uv = s_1 \dots s_n t_1 \dots t_m$. Note that $(W(A), *)$ has a structure of monoid with $epsilon$ as the identity element.

Definition 2.2. $(W(A), *)$ *is the* ***free monoid*** *on A. A monoid M is said to be* ***free*** *if it contains a subset $S \subset$ such that M is isomorphic to $W(S)$.*

Example. $(\mathbb{N}, +)$ is isomorphic to the free monoid on one element $W(\{a\})$. $0 \in \mathbb{N}$ represents the empty word ϵ; $1 \in \mathbb{N}$ represents the word a; and every $n \in \mathbb{N}$ represents the only word $aa \dots a$ of length n.

Exercise. Let A be a set with $|A| = m$, $n \geq 0$ be an integer. Show that the number of words of length n is m^n.

2.1.2 Free groups and reduced words

The essential difference between a group and a monoid is the existence of *inverse elements*. Let S be a non-empty set of symbols. Denote by $S^{-1} := \{s^{-1} \mid s \in S\}$ the set of formal inverse symbols of S, with the convention that $(s^{-1})^{-1} = s$.

Thus, for any $n \in \mathbb{Z}$ and $s \in S \cup S^{-1}$, we write s^n for the word $s \ldots s$ (n times) if $n > 0$; or $s^{-1} \ldots s^{-1}$ ($-n$ times) if $n < 0$; or ϵ (the empty word) if $n = 0$.

Definition 2.3. *A word $w \in W(S \sqcup S^{-1})$ is said to be* **reduced** *if it has no subword of the form ss^{-1} for some $s \in S \sqcup S^{-1}$.*

Example. Let $S = \{a, b, c\}$. Consider the words in $W(a, b, c, a^{-1}, b^{-1}, c^{-1})$.

- The word $abac^{-1}$ is reduced.
- The word $cb^{-1}aa^{-1}bc$ is not reduced.

A **reduction** on a non-reduced word $w \in W(S \sqcup S^{-1})$ is the cancellation of some subword of the form ss^{-1} for some $s \in S \sqcup S^{-1}$.

Example. The empty word ϵ is a reduction of the word aa^{-1}.

Theorem 2.4. *Let $w \in W(S \sqcup S^{-1})$. There exists a unique reduced word w' that is obtained by a sequence of reductions on w, called the* **reduced form** *of w.*

To prove the theorem we need the following lemma.

Lemma 2.5. *Let $w \in W(S \sqcup S^{-1})$ be a non-reduced word and let $w_1 \neq w_2 \in W(S \sqcup S^{-1})$ obtained from w by one reduction. Then there exists a word u that can be obtained by one reduction from both w_1 and w_2.*

Proof. Let $w = s_1, \ldots, s_n$ with $s_i \in S \sqcup S^{-1}$. Suppose that the reduction on w_1 occurs at $s_i s_{i+1}$ and the reduction on w_2 occurs at $s_j s_{j+1}$. If $i = j$ then $w_1 = w_2$, contradiction. If $|i - j| = 1$, then either $s_i s_{i+1} s_{i+2}$ or $s_j s_{j+1} s_{j+2}$ has the form $ss^{-1}s$ for some $s \in S \sqcup S^{-1}$. In either case, only s will be left after the cancellation, which leads to $w_1 = w_2$, contradiction.

Hence, $|i - j| \geq 2$. One can then obtain u by canceling the subword $s_j s_{j+1}$ of w_1 or the subword $s_i s_{i+1}$ of w_2. □

Proof of the theorem. Let $w \in W(S \sqcup S^{-1})$. We shall prove the theorem by induction on the length of w. First, the theorem holds for words of length 0 or 1 since they are always reduced. Now let w be a word of length $n \geq 2$ and assume that the theorem holds for all words of length at most $n - 1$.

Existence. Each reduction decreases the length of w by 2. By induction, one can obtain a reduced word from w by a finite sequence of reductions.

Uniqueness. If w is reduced, then it is clearly the unique reduced word obtained by any sequence of reductions applied to w. If there is a unique word u obtained by a single reduction of w, then by the inductive hypothesis there exists a unique reduced word u' obtained by any sequence of reductions applied to u. Since every sequence of reductions applied to w must pass through u, it follows that u' is the unique reduced word obtained from w.

Assume now that there exist two distinct words w_1 and w_2, both obtained by a single reduction of w. By the lemma, there exists a common word u for w_1 and w_2 obtained by one reduction. By the inductive hypothesis, there exist unique reduced words w_1', w_2', and u' obtained by sequences of reductions applied to w_1, w_2, and u, respectively.

The sequence $w_1 \to u \to u'$ yields a sequence of reductions applied to w_1. By uniqueness, we must have $u' = w_1'$. Similarly, $u' = w_2'$. Hence $w_1' = w_2'$.

Since this argument applies to any pair of distinct words obtained by a single reduction of w, it follows that there is a unique reduced word obtained by a sequence of reductions applied to w. □

We can thus define an equivalence relation $\sim$ on $W(S \sqcup S^{-1})$ by declaring that $u \sim v$ if and only if u and v have the same reduced form. Let $F(S) = W(S \sqcup S^{-1})/\sim$, which can be identified with the set of reduced words on $S \sqcup S^{-1}$. Define an operation on $F(S)$ by declaring $u \cdot v$ to be the reduced form of $u * v \in W(S \sqcup S^{-1})$.

Exercise. Show that $(F(S), \cdot)$ is a group.

Definition 2.6. *$(F(S), \cdot)$ is the **free group** on S. A group G is said to be **free** if it contains a subset $S \subset$ such that G is isomorphic to $F(S)$.*

We will abuse notation by identifying elements of $F(S)$ with their representatives in $W(S \sqcup S^{-1})$. For instance, with $S = \{a, b\}$, we may write $g = aba^{-1} \in F(S)$. There are thus some *equations* in $F(S)$ such as $abb^{-1}a = aa$. By convention, $1_{F(S)}$ denotes the identity element of $F(S)$, which is represented by the empty word, as well as any word of the form $s_1 \dots s_n s_n^{-1} \dots s_1^{-1}$. For any $s \in S \sqcup S^{-1}$ and $n \in \mathbb{N}$, we write $s^n = s \dots s$ (n times) and $s^{-n} = s^{-1} \dots s^{-1}$ (n times), with the convention that $s^0 = 1_{F(S)}$.

The **canonical injection** $i\colon S \hookrightarrow F(S)$ sends each $s \in S$ to the reduced word s. Via this inclusion, we identify S as a subset of $F(S)$. Clearly, S is a generating set of $F(S)$. The **word length** of an element $g \in F(S)$ with respect to S is then exactly the length of its reduced form, that is, the minimal length of a word in $W(S \sqcup S^{-1})$ representing g.

Example. $(\mathbb{Z}, +)$ is isomorphic to the free group on one element $F(\{a\})$, with $n \in \mathbb{Z}$ identified with a^n.

Exercise. Let S be a set with $|S| = m$, let $n \geq 1$ be an integer. Show that $F(S)$ has $2m(2m-1)^{n-1}$ elements of length n.

Proposition. The Cayley graph $\mathrm{Cay}(F(S), S)$ is a tree. If S is finite, it is a $2|S|$-regular tree.

Proposition. Let G be a group with a finite generating set S such that $1 \notin S$. If $S \cap S^{-1} = \varnothing$, then:

- There is a one-to-one correspondence between paths in $\mathrm{Cay}(G, S)$ starting at 1_G and the set of words $W(S \sqcup S^{-1})$.
- There is a one-to-one correspondence between cycles in $\mathrm{Cay}(G, S)$ based at 1_G and the set of words on $S \cup S^{-1}$ that are equal to the identity in G.
- There is a one-to-one correspondence between paths in $\mathrm{Cay}(G, S)$ starting at 1_G *without backtracking* (i.e. containing no subpath of the form (u, v, u) for some edge $\{u, v\}$) and the set of reduced words $F(S)$.
- There is a one-to-one correspondence between cycles in $\mathrm{Cay}(G, S)$ based at 1_G *without backtracking* and reduced words on $S \cup S^{-1}$ that are equal to the identity in G.

Theorem 2.7. *A group is free if an only if there exists a tree on which it acts freely.*

Proof. Homework. □

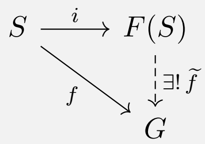
Corollary 2.8. *Every subgroup of a free group is free.*

2.1.3 Universal property

Theorem 2.9 (Universal property). *The free group $F(S)$ together with the canonical inclusion $i\colon S \hookrightarrow F(S)$ satisfies the following* ***universal property****:*
For every group G and every map $f : S \to G$, there exists a unique group homomorphism

$$\widetilde{f} : F(S) \to G$$

that extends f. That is, the following diagram ***commutes****, giving $\widetilde{f} \circ i = f$.*



Proof. **Existence:** Given a map $f : S \to G$. Construct $\widetilde{f} : F(S) \to G$ by $\widetilde{f}(s) := f(s)$ for any $s \in S$, $\widetilde{f}(s^{-1}) := f(s)^{-1}$ for any $s \in S$, and

$$\widetilde{f}(s_1 s_2 \dots s_n) := \widetilde{f}(s_1)\widetilde{f}(s_2)\dots\widetilde{f}(s_n)$$

for any word $s_1 \dots s_n \in W(S \sqcup S^{-1})$.

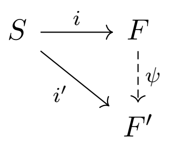
Now we need to check that $\widetilde{f}$ is a well-defined group homomorphism. Note that by construction, if v is a word obtained from w by one reduction, then $\widetilde{f}(w) = \widetilde{f}(v)$ since $\widetilde{f}(ss^{-1}) = \widetilde{f}(s)\widetilde{f}(s)^{-1} = 1_G$. Hence, if w_1, w_2 are two words in $W(S \sqcup S^{-1})$ that are equal in $F(S)$, since $w_1' = w_2'$ (the reduced form, by Theorem 2.4), we have $\widetilde{f}(w_1) = \widetilde{f}(w_1') = \widetilde{f}(w_2') = \widetilde{f}(w_2)$. So the image of an element in $F(S)$ by $\widetilde{f}$ is independent of its word representative. Hence $\widetilde{f}$ is well-defined.

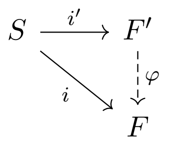
The map is clearly a group homomorphism by its construction.
Uniqueness: If $\widetilde{f_1} \circ i = f = \widetilde{f_2} \circ i$, then $\widetilde{f_1} = \widetilde{f_2}$ since they agree on the generating set S of $F(S)$. □

Proposition. Let G be a group with a generating set $S \subset G$. Then G is a quotient of $F(S)$. In particular, every group is a quotient of a free group.

Theorem 2.10. *Let S be a set. Let F and F' be groups with inclusion maps $i\colon S \hookrightarrow F$ and $i'\colon S \hookrightarrow F'$. If (F, i) and (F', i') both satisfy the universal property, then F and F' are isomorphic.*

Proof. Apply the universal property of (F, i) to (F', i'). There exists a unique homomorphism $\psi : F \to F'$ that extends i'. Similarly, by the universal property of (F', i') applied to (F, i), there is a unique homomorphism $\varphi : F' \to F$ that extends i.





Thus $\varphi \circ \psi\colon F \to F$ is a group homomorphism that extends i, because $\varphi \circ \psi \circ i = \varphi \circ i' = i$. By the universal property of (F, i) applied to itself, id_F is the unique homomorphism that extends i. Hence

$$\varphi \circ \psi = \mathrm{id}_F.$$

Similarly,

$$\psi \circ \varphi = \mathrm{id}_{F'}.$$

Thus F is isomorphic to F'. □

Corollary 2.11. *Let S be a set, F be a group, and $i \hookrightarrow S \to F$. If (F, i) satisfies the universal property, then F is isomorphic to $F(S)$.*

2.1.4 Ping-pong lemma

Theorem 2.12 (Ping-pong lemma)**.** *Let G be a group generated by 2 elements a and b. Suppose that G acts on a set X such that there are non-empty subsets $A, B \subset X$ with $B \not\subset A$ satisfying*

$$a^n \cdot B \subset A \quad \text{and} \quad b^n \cdot A \subset B \quad \forall n \in \mathbb{Z}\backslash\{0\},$$

then G is free of rank 2, generated by $\{a, b\}$.

Proof. It suffices to show that every non-empty reduced word on $\{a, b\}$ is not the identity element in G.

Let w be a non-empty reduced word. Suppose that w starts and ends by some power of a. Otherwise, we can replace w by $w' = a^n w a^{-n}$ for some n large enough, since w is the identity in G if and only of w' is the identity in G. Now we write

$$w = a^{n_1} b^{m_1} a^{n_2} b^{m_2} \dots a^{n_k} b^{m_k} a^{n_{k+1}}$$

where all the n_i, m_i are non zero integers. Then we play ping-pong:

$$\begin{aligned} w \cdot B &= a^{n_1} b^{m_1} a^{n_2} b^{m_2} \dots a^{n_k} b^{m_k} a^{n_{k+1}} \cdot B \\ &\subset a^{n_1} b^{m_1} a^{n_2} b^{m_2} \dots a^{n_k} b^{m_k} \cdot A \\ &\subset a^{n_1} b^{m_1} a^{n_2} b^{m_2} \dots a^{n_k} \cdot B \\ &\subset \qquad \vdots \\ &\subset a^{n_1} b^{m_1} \cdot A \\ &\subset a^{n_1} \cdot B \subset A \end{aligned}$$

But B is not included in A, so w can not be the identity element. □

Application: free linear groups.

Proposition. The subgroup G of $GL_2(\mathbb{R})$ generated by

$$a = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

is free of rank 2.

Proof. Consider the action of G on $\mathbb{R}^2$ by matrix multiplication. Take the subsets

$$A = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, |x| > |y| \right\} \quad \text{and} \quad B = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, |x| < |y| \right\}.$$

Then we play ping-pong. □

2.2 Group presentation

In the last subsection we have seen that every group G is a quotient of some free group $F(S)$. We can describe G from $F(S)$ by a *group presentation*, which is a convenient way to describe a group.

2.2.1 Generators and relations

Exercise. Let G be a group. The intersection of a collection of normal subgroups of G is a normal subgroup of G.

> **Definition 2.13.** *Let G be a group and let $R \subset G$. The* ***normal subgroup*** *of G* ***generated*** *by R, denoted by $\langle\langle R\rangle\rangle$, is the smallest normal subgroup of G that contains R.*

Equivalently, $\langle\langle R\rangle\rangle$ is the intersection of all the normal subgroups of G that contains R.

Proposition. Let G be a group and R be a subset. We denote $R^{-1} := \{r^{-1} \mid r \in R\}$. Then

$$\langle\langle R\rangle\rangle = \left\{ \prod_{i=1}^{n} a_i r_i a_i^{-1} \,\middle|\, n \in \mathbb{N}, r_i \in R \cup R^{-1}, a_i \in G \right\}.$$

> **Definition 2.14.** *Let S be a set, $F(S)$ be the free group on S. Let R be a subset of $F(S)$. We say that a group G admits the* ***group presentation***
>
> $$\langle S \mid R \rangle,$$
>
> *where S is called the set of* ***generators*** *and R is called the set of* ***relators****, if*
>
> $$G \cong F(S)/\langle\langle R\rangle\rangle.$$

We often write $G = \langle S \mid R\rangle$ or $G \cong \langle S \mid R\rangle$ if G admits the presentation $\langle S \mid R\rangle$.

Example. $\mathbb{Z}/n\mathbb{Z} = \langle a \mid a^n\rangle$ (why? exercise!)

Example. $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1}\rangle$ (need a proof, see the end of the section)

Example. $D_n = \langle r, s \mid r^n, s^2, rsrs\rangle$ (need a proof, see the end of the section)

Example. $F(S) = \langle S \mid \varnothing\rangle$

For every word $r \in R \subset F(S)$ in the presentation $G = \langle S \mid R\rangle$, the equality $r = 1$ holds in G; such an equality is called a **relation**. By the previous proposition, *any product of conjugates of relations is also trivial* in G. In other words,

$$\prod_{i=1}^{n} a_i r_i a_i^{-1} = 1 \quad \text{in} \quad G$$

for all $n \in \mathbb{N}$, $r_i \in R \cup R^{-1}$, and $a_i \in F(S)$.

For example, consider the presentation

$$D_n = \langle r, s \mid r^n, s^2, rsrs\rangle.$$

Since $r^n = 1$ in D_n, conjugating by s gives

$$sr^n s = sr^n s^{-1} s^2 = 1$$

in D_n.

2.2.2 Finitely presented groups

Given a group G with a finite generating set S. By the universal property of free groups applying on the canonical inclusion $S \hookrightarrow G$, there is a unique group morphism

$$\varphi : F(S) \to G$$

that maps $s \in S$ to $s \in G$ for any s. As S is a generating set, φ is surjective. Let $N = \ker(\varphi)$, we can conclude that for any subset $R \subset F(S)$ such that $N = \langle\langle R \rangle\rangle$, we have

$$G = \langle S \mid R \rangle \,.$$

A group may have several different presentations. A group is called **finitely presented** if it admits a presentation with a finite set of generators and a finite set of relations. Finitely presented groups are finitely generated, but finitely generated groups that are not necessarily finitely presented.

Example. $\mathbb{Z} = \langle a, b \mid ab^2 \rangle$

Example. $\{1\} = \langle a \mid a \rangle = \langle a, b \mid ab, ab^2 \rangle$

Example. The lamplighter group

$$\mathbb{Z}_2 \wr \mathbb{Z} \ = \ \left\langle a, t \;\middle|\; a^2 = 1,\ [a, t^k a t^{-k}] = 1 \text{ for all } k \in \mathbb{Z} \right\rangle,$$

where the bracket notation means $[a, b] = aba^{-1}b^{-1}$, is infinitely presented.

Theorem 2.15. *A group is finitely presented if and only if it has an isometric action on a* simply connected *geodesic metric space that is proper and cobounded.*

Proof. Too difficult. See [Bridson-Haefliger] p. 137, Corollary 8.11. ☐

2.2.3 Universal property

Theorem 2.16 (Universal property)**.** *The presentation $\langle S \mid R \rangle$ together with the canonical map $i \colon S \to \langle S \mid R \rangle$ (which need not be injective) satisfies the following* ***universal property****:*
For every group G and every map $f : S \to G$ such that, for every relator

$$r = s_1^{\epsilon_1} \dots s_n^{\epsilon_n} \in R, \quad s_i \in S,\, \epsilon_i \in \{\pm 1\},$$

one has

$$f(s_1)^{\epsilon_1} \dots f(s_n)^{\epsilon_n} = 1 \quad \text{in } G,$$

there exists a unique group homomorphism

$$\varphi : \langle S \mid R \rangle \to G$$

that extends f. That is, the following diagram ***commutes****, giving $\varphi \circ i = f$.*

$$\begin{array}{ccc} S & \xrightarrow{\ i\ } & \langle S \mid R \rangle \\ & \underset{f}{\searrow} & \Big\downarrow {\scriptstyle \exists! \varphi} \\ & & G \end{array}$$

Proof. **Existence.** Let G be a group and let $f\colon S \to G$ be a map that satisfies the conditions of the theorem. By the universal property of the free group, f extends uniquely to a homomorphism

$$\widetilde{f}\colon F(S) \to G.$$

Since each relator $r \in R$ is mapped to the identity by $\widetilde{f}$, the normal closure $\langle\langle R\rangle\rangle$ is contained in $\ker(\widetilde{f})$. Hence $\widetilde{f}$ factors through the quotient

$$\pi\colon F(S) \to F(S)/\langle\langle R\rangle\rangle = \langle S \mid R\rangle$$

inducing a homomorphism

$$\varphi\colon \langle S \mid R\rangle \to G$$

such that $\varphi \circ \pi = \widetilde{f}$. In particular, $\varphi \circ i = f$.

Uniqueness. Suppose $\varphi_1, \varphi_2\colon \langle S \mid R\rangle \to G$ are homomorphisms such that

$$\varphi_1 \circ i = \varphi_2 \circ i = f.$$

Then φ_1 and φ_2 agree on the image of S, which generates $\langle S \mid R\rangle$ as a group. Therefore $\varphi_1 = \varphi_2$. □

Proposition. Let $G = \langle a, b \mid aba^{-1}b^{-1}\rangle$. Prove that $G \cong \mathbb{Z}^2$ where $\mathbb{Z}^2$ is regarded as the group direct product $\mathbb{Z} \times \mathbb{Z} := \{(m, n) \mid m, n \in \mathbb{Z}\}$.

Proof. By the universal property, there is a group epimorphism

$$\varphi\colon G \to \mathbb{Z}^2$$

such that $\varphi(a) = (1, 0)$ and $\varphi(b) = (0, 1)$. Now we have to prove that φ is a monomorphism.

The equation $aba^{-1}b^{-1}$ deduces that $ab = ba$, $a^{-1}b^{-1} = a^{-1}b^{-1}$, $ab^{-1} = b^{-1}a$ and $a^{-1}b = ba^{-1}$. So in the group G, every word $w \in W(\{a, b\})$ can be written as the form $a^m b^n$ for some $m, n \in \mathbb{Z}$.

Let $w = a^m b^n$ be an element of G such that $\varphi(w) = (0, 0)$. Then

$$\varphi(w) = \varphi(a^m b^n) = m\varphi(a) + n\varphi(b) = (m, n),$$

so $(m, n) = (0, 0)$, which implies that $w = a^0 b^0 = 1$. □

Exercise. Show that $D_n \cong G := \langle r, s \mid r^n, s^2, rsrs\rangle$.

Hint: Use the universal property, write every element of G in the form $r^i s^j$ for $0 \leq i \leq n-1$ and $0 \leq j \leq 1$, then conclude by the cardinalities of D_n and of G.

3 Quasi-geometry

We have seen that a finitely generated group G can be regarded as a metric space (G, d_S) (or as its Cayley graph $\mathrm{Cay}(G, S)$), where S is a finite generating set and d_S is the associated word metric. However, this construction depends on the choice of the finite generating set S. Thus, at this stage, the "geometry" of the group appears to depend on that choice.

The goal of this section is to show that for any two finite generating sets S and S' of G, the metric spaces (G, d_S) and $(G, d_{S'})$ are equivalent in an appropriate sense, namely in the sense of *quasi-isometry*.

3.1 Quasi-isometry

3.1.1 Definitions

Definition 3.1 (Quasi-isometry)**.** *Let $f : (X, d_X) \to (Y, d_Y)$ be a map between metric space.*

- *The map f is a* ***quasi-isometric embedding*** *if there exist $a \geq 1, c \geq 0$ such that for any $x, x' \in X$,*
$$\frac{1}{a}(d_X(x, x')) - c \leq d_Y(f(x), f(x')) \leq a d_X(x, x') + c.$$
- *The map f is* ***quasi-dense*** *if there exists $r > 0$ such that for any $y \in Y$ there exists $x \in X$ such that $d_Y(f(x), y) < r$.*
- *The map f is a* ***quasi-isometry*** *if it is a quasi-dense quasi-isometric embedding. We say that the metric spaces X, Y are* ***quasi-isometric****.*

  We denote $X \sim_{QI} Y$.

Example. Some examples of quasi-isometries.

- The canonical injection from $\mathbb{Z}^2$ to $\mathbb{R}^2$.
- The floor map $\mathbb{R} \to \mathbb{Z}$ defined by $x \mapsto \lfloor x \rfloor$.
- Any non-degenerated affine transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Example. Some examples of quasi-isometric embeddings that are not quasi-dense.

- The canonical injection from $\mathbb{Z}$ to $\mathbb{R}^2$.
- The canonical injection from the free group $F(a, b)$ to $F(a, b, c)$.

Exercise. Give some examples of quasi-dense maps that are not quasi-isometric embeddings.

Theorem 3.2. *Quasi-isometry is an equivalence relation.*

Proof. Transitivity and reflexive are easy. Let us prove the symmetry. We denoted X, Y instead of $(X, d_X), (Y, d_Y)$.

Let $f : X \to Y$ be a quasi-dense quasi-isometric embedding, with constants r, a, c as given in the definition. We shall construct an "quasi-inverse map" $g : Y \to X$ that has the same properties.

For each $y \in Y$, we define $g(y)$, by the axiom of choice, as one of the $x \in X$ such that $d(f(x), y) \leq r$. That is, $d(f(g(y)), y) \leq r$ for every $y \in Y$.

By the first part of the q.-i. embedding inequality of f, for any $x \in X$, we have

$$d_X(x, g(f(x))) \leq a d_Y(f(x), f(g(f(x)))) + ac \leq ar + ac.$$

Hence g is quasi-dense.

Now we prove that g is a quasi-isometric embedding. Let $y, y' \in Y$. By triangle inequality,

$$d_Y(y,y') \leq d_Y(f(g(y)), f(g(y'))) + d_Y(y, f(g(y))) + d_Y(y', f(g(y'))) \leq d_Y(f(g(y)), f(g(y'))) + 2r.$$

By the second part of the q.-i. embedding inequality of f,

$$d_Y(y,y') \leq a d_X(g(y), g(y')) + c + 2r = a d_X(g(y), g(y')) + c + 2r.$$

Hence,

$$\frac{1}{a} d_Y(y,y') - \frac{c+2r}{a} \leq d_X(g(y), g(y')).$$

Similarly, by triangle inequality,

$$d_Y(y,y') \geq d_Y(f(g(y)), f(g(y'))) - d_Y(y, f(g(y))) - d_Y(y', f(g(y'))) \geq d_Y(f(g(y)), f(g(y'))) - 2r.$$

We get, by the first part of the q.-i. embedding inequality,

$$d_X(g(y), g(y')) \leq a d_Y(f(g(y)), f(g(y'))) + ac \leq a d_Y(y,y') + 2ar + ac.$$

Therefore, g is a quasi-isometric embedding. □

Example. Bounded spaces are quasi-isometric to each other.

Example. Let G be a group with S a finite generating set. Let $H < G$ be a finite index subgroup. Then the canonical injection $(H, d_S) \to (G, d_S)$ is a quasi-isometry.

3.1.2 Quasi-isometry and groups: Švarc-Milnor lemma

The following result is sometimes called the "fundamental lemma of Geometric Group Theory".

Theorem 3.3 (Švarc-Milnor lemma)**.** *Let G be a group that acts on a geodesic metric space (X, d). If the action is isometric, proper, and cobounded, then*

(i) *G is finitely generated.*

(ii) *For any finite generating set S of G, (G, d_S) is quasi-isometric to (X, d).*

Proof. (i) was proved at the end of Section 1. We now prove (ii).

Let S be a finite generating set of G, and let d_S denote the associated word metric. Fix a basepoint $x_0 \in X$, and consider the orbit map

$$\begin{aligned} \Phi\colon G &\longrightarrow X \\ g &\longmapsto g \cdot x_0. \end{aligned}$$

Φ is clearly quasi-dense by coboundedness. Now we prove that Φ is a quasi-isometric embedding. Set

$$L = \max_{s \in S} d(x_0, s \cdot x_0).$$

For any $g, h \in G$, write $g^{-1}h = s_1 \cdots s_n$ with $n = d_S(g,h)$. Using the triangle inequality and the G–invariance of d, we obtain

$$\begin{aligned} d(\Phi(g), \Phi(h)) &= d(g \cdot x_0, h \cdot x_0) \\ &= d(x_0, g^{-1}h \cdot x_0) \\ &\leq \sum_{i=1}^{n} d(x_0, s_i \cdot x_0) \\ &\leq Ln \\ &= L\, d_S(g,h). \end{aligned}$$

To prove the other inequality, we use the construction from the proof of (i). Choose $r > 0$ such that $G \cdot B(x_0, r) = X$. As shown previously, the set

$$T = \{g \in G \mid g \cdot x_0 \in B(x_0, 3r)\}$$

is a finite generating set of G, and we denote by d_T the corresponding word metric.

Moreover, if $g \in G$ satisfies $(n-1)r \leq d(x_0, g \cdot x_0) \leq nr$, then there exist elements $t_0, \dots, t_{n-1} \in T$ such that

$$g = t_{n-1} \cdots t_0.$$

Consequently,

$$d(x_0, g \cdot x_0) \geq r\, d_T(e, g) - r.$$

Now define

$$M = \max_{t \in T} d_S(e, t).$$

If $g \in G$ with $d_T(e, g) = n'$ and $g = t'_1 \cdots t'_{n'}$ is a shortest word in T, then by the triangle inequality,

$$d_S(e, g) \leq \sum_{i=1}^{n'} d_S(e, t'_i) \leq Mn' = M\, d_T(e, g).$$

Combining the previous inequalities, for any $g, h \in G$ we obtain

$$d_S(g, h) = d_S(e, g^{-1}h) \leq \frac{M}{r} d(x_0, g^{-1}h \cdot x_0) + M = \frac{M}{r} d(\Phi(g), \Phi(h)) + M.$$

Therefore, Φ is a quasi-isometric embedding. $\square$

Corollary 3.4. *Let G be a finitely generated group and let S, S' be two different finite generating sets. Then (G, d_S) and $(G, d_{S'})$ are quasi-isometric.*

Exercise. Prove the corollary above without using Milnor-Švarc lemma.

Definition 3.5. *Two finitely generated groups G_1, G_2 are said to be quasi-isometric if the metric spaces (G_1, d_{S_1}), (G_1, d_{S_1}) are quasi-isometric with respect to any finite generating sets S_1, S_2. We denote $G_1 \sim_{QI} G_2$.*

Proposition. Let H be a finite index subgroup of a finitely generated group G. Then H is finitely generated and quasi-isometric to G.

Proof. Consider the (left) action of H on $\mathrm{Cay}(G, S)$, where S is a finite generating set of G. This action is free, hence proper.

Let $n = [G : H]$, and choose representatives $g_1, \dots, g_n$ of the *right* cosets in G/H such that each g_i has minimal word length in its coset Hg_i. Set

$$r = \max_{1 \leq i \leq n} |g_i|.$$

Then every element $g \in G$ can be written as $g = hg_i$ for some i and some $h \in H$, with $|g_i| \leq r$. It follows that

$$H \cdot B(\mathrm{id}_G, r)$$

covers $\mathrm{Cay}(G, S)$, so the action is cobounded.

Consequently, H is finitely generated and quasi-isometric to $\mathrm{Cay}(G, S)$. $\square$

Corollary 3.6. *Finite index subgroups of a finitely generated group are quasi-isometric.*

3.2 Growth

Isomorphic groups are of course quasi-isometric, but the inverse is not true in general, since any finite index subgroup is quasi-isometric to its ambient group. Properties of groups that are invariant under quasi-isometries are called **quasi-isometric invariants**. More precisely, a group property P is a quasi-isometric invariant of $G_1 \sim_{QI} G_2$ and G_1 satisfies P implies that G_2 satisfies P.

The **growth** of a group is a quasi-isometric invariant.

In the following, we adopt the convention that $\mathbb{N}$ include 0 and $\mathbb{R}^+$ include 0.

3.2.1 Growth of a group

Definition 3.7. *Let G be a finitely generated group with S a finite generating set. The* ***growth function*** *of G with respect to S is the function*

$$\begin{aligned} \beta_{G,S} \colon \mathbb{N} &\to \mathbb{N} \\ r &\mapsto |B_{(G,d_S)}(e_G, r)|. \end{aligned}$$

Example. Some examples of growths.

- $\beta_{\mathbb{Z},\{1\}}(r) = 2r + 1 = O(r)$.
- $\beta_{\mathbb{Z}^2,\{(1,0),(0,1)\}}(r) = 2r^2 + 2r + 1 = O(r^2)$.
- Let S be a set with $|S| = n$. Then $\beta_{F(S),S}(r) = 1 + \frac{n}{n-1}\left((2n-1)^r - 1\right) = O((2n-1)^r)$.
- Let G be a finite groups, then for any r larger than the diameter of (G, d_S), $\beta_{G,S}(r) = |G| = O(1)$.

Proposition. Let G be a group with a finite generating set S.

1. If G is infinite, then the growth function is strictly increasing. In particular, for any $r \in \mathbb{N}$,
$$\beta_{G,S}(r) \geq r.$$
2. The growth function is sub-multiplicative. In other words, for any $r, r' \in \mathbb{N}$,
$$\beta_{G,S}(r + r') \leq \beta_{G,S}(r)\beta_{G,S}(r').$$
3. The growth of G is always slower than the free group on S. That is, for any $r \in \mathbb{N}$,
$$\beta_{G,S}(r) \leq \beta_{F(S),S}(r).$$

3.2.2 Growth equivalent functions

Definition 3.8. *Let $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ be increasing functions. We say that f is* ***dominated*** *by g, or that g dominates f, and write $f \prec g$, if there exist constants $a, b \geq 1$, $c, d \geq 0$ and $x_0 \in \mathbb{R}^+$ such that for every $x \geq x_0$,*

$$f(x) \leq a\, g(bx + c) + d.$$

If $f \prec g$ and $g \prec f$, we say that f and g are ***growth equivalent****, or that they have equivalent growth. In this case we write*

$$f \asymp g.$$

Exercise. Show that g dominate f if and only if there exists $c > 0$ such that for every $x \in \mathbb{R}^+$, we have

$$f(x) \leq cg(cx + c) + c.$$

Proposition. Being growth equivalent is an equivalence relation.

Proposition. The relation "$\prec$" defines a partial order on the set of increasing functions from $\mathbb{R}^+$ to $\mathbb{R}^+$, but *not a total order*.

Exercise. Find a pair of functions f, g such that $f \not\prec g$ and $g \not\prec f$.

Example. For $\alpha \geq 0$, define

$$\begin{array}{rcl} f_\alpha \colon \mathbb{R}^+ & \to & \mathbb{R}^+ \\ x & \mapsto & x^\alpha. \end{array}$$

Then $f_\alpha \prec f_\beta$ if and only if $\alpha \leq \beta$.
Consequently, $f_\alpha \asymp f_\beta$ if and only if $\alpha = \beta$.

Example. Let P and Q be polynomials with positive coefficients. Then $P \prec Q$ if and only if $\deg(P) \leq \deg(Q)$. Consequently, $P \asymp Q$ if and only if $\deg(P) = \deg(Q)$.

Example. For $\alpha > 1$, define

$$\begin{array}{rcl} g_\alpha \colon \mathbb{R}^+ & \to & \mathbb{R}^+ \\ x & \mapsto & \alpha^x. \end{array}$$

Then $g_\alpha \asymp g_\beta$ for any $\alpha, \beta > 1$.
Therefore, for any $\alpha, \alpha' > 1$ and $\beta \geq 0$,

$$(x \mapsto x^\beta \alpha^x) \asymp g_{\alpha'}$$

Theorem 3.9. *Let G and H be finite generating groups with finite generating sets S and T, respectively.*

1. *If there is a quasi-isometric embedding from (G, d_S) to (H, d_T), then*

$$\beta_{G,S} \prec \beta_{H,T}.$$

2. *If G and H are quasi-isometric, then*

$$\beta_{G,S} \asymp \beta_{H,T}.$$

Proof. First, as the action of G on itself by left multiplication is isometric, for every $x \in G$ and $r > 0$ we have $|B_{G,S}(x, r)| = \beta_{G,S}(r)$. The same applies for H.

Let $f : G \to H$ be a quasi-isometric embedding. Let $a \geq 1, c \geq 0$ such that for any $x, y \in G$,

$$\frac{1}{a} d_S(x, y) - c \leq d_T(f(x), f(y)) \leq a d_S(x, y) + c.$$

Using the first part of the inequality, if $f(x) = f(y)$ then $d_S(x, y) \leq ac$. So the preimage of $f(x) \in H$ has at most $|B_{G,S}(x, ac)| = \beta_{G,S}(ac)$ elements.

Now fix $x \in G$ and $r > 0$. Using the second part of the inequality, for every $y \in B_{G,S}(x, r)$, we have $f(y) \in B_{H,T}(f(x), ar + c)$. Therefore,

$$f(B_{G,S}(x, r)) \subset B_{H,T}(f(x), ar + c).$$

Hence, for every $r > 0$,

$$|B_{G,S}(x,r)| \leq \beta_{G,S}(ac) \times |B_{H,T}(f(x), ar + c)|,$$

which implies that

$$\beta_{G,S}(r) \leq \beta_{G,S}(ac) \times \beta_{H,T}(ar + c).$$

As the term $\beta_{G,S}(ac)$ does not depend on the radius r, we get

$$\beta_{G,S} \prec \beta_{H,T}.$$

The second assertion follows directly from the first one. □

The theorem tells us that the growth equivalence is a quasi-isometric invariance. In particular, for any finite generating sets S, S' of a finitely generated group G, we have

$$\beta_{G,S} \asymp \beta_{G,S'}.$$

Therefore, the following definition makes sense.

Definition 3.10. *The* ***growth type*** *of a finitely generated group G is the (common) growth equivalence class of the growth function $\beta_{G,S}$ for* any *finite generating set S of G. We denote the growth type of G by*

$$\beta_G,$$

which is an equivalence class of functions.

We will still use the relations $\prec$ and $\asymp$ for equivalent classes of functions. The following corollary is straightforward.

Corollary 3.11. *The growth type of a group is a quasi-isometric invariant.*
That is, if G and H are finitely generated groups that are quasi-isometric, then

$$\beta_G \asymp \beta_H.$$

Consequently, finitely generated groups with different growth types can not be quasi-isometric.

Proposition. Let G be a finitely generated group and H be a finitely generated subgroup of G. Then

$$\beta_H \prec \beta_G.$$

Proof. Let S be a finite generating set of G and T be a finite generating set of H. Then, $S' = S \cup T$ is a finite generating set of G.

For every $x \in H$, because $T \subset S'$, we have $d'_S(e,x) \leq d_T(e,x)$. So for every $r > 0$, we have

$$B_{H,T}(e,r) \subset B_{G,S'}(e,r).$$

In particular,

$$\beta_{H,T} \prec \beta_{G,S'} \asymp \beta_{G,S}.$$

□

3.3 Some big theorems

In this subsection we discuss two major results on the growth of groups: one due to Grigorchuk (1980, 1984) and another due to Gromov (1981). No proofs will be given here.

See [de la Harpe, Chapter VIII] for a proof of Grigorchuk's theorem; and [Druţu-Kapovich, Chapter 16] for a proof of Gromov's theorem.

3.3.1 Grigorchuk's Theorem

Definition 3.12. *Let G be a finitely generated group with a generating set S.*

1. *We say that G has* exponential growth *if $\beta_{G,S} \asymp (x \mapsto \alpha^x)$ for some (any) $\alpha > 1$.*
2. *We say that G has* polynomial growth *if $\beta_{G,S} \prec (x \mapsto x^\alpha)$ for some $\alpha \geq 1$.*
3. *We say that G has* intermediate growth *if G has nether exponential nor polynomial growth.*

Example. Some examples of growth types.

1. For a finite group G with any generating set S, we have $\beta_{G,S} \asymp 1$ (the constant function).
2. Finitely generated abelian groups have polynomial growths.
3. Finitely generated free groups have exponential growth.

Theorem 3.13 (Grigorchuk)**.** *There exist finitely generated groups of intermediate growth. More precisely, the examples known so far have growth types equivalent to*

$$x \mapsto e^{x^\alpha}$$

for some $0 < \alpha < 1$.

3.3.2 Gromov's Theorem

Theorem 3.14 (Gromov)**.** *A finitely generated group has polynomial growth if and only if it is **virtually nilpotent**.*

More precisely, if G is nilpotent we know how to compute its growth (see the last Theorem of this section). And, if G has polynomial growth, then the growth is $x \mapsto x^d$ with some *integer* $d \geq 1$.

Corollary 3.15. *Being* virtually nilpotent *is a quasi-isometric invariant. That is, if two groups are quasi-isometric and one of them is virtually nilpotent, then the other one is also virtually nilpotent.*

Now we will explain what do "virtually" and "nilpotent" mean.

Virtual properties.

Definition 3.16. *Let $\mathcal{P}$ be a property of groups. We say that a group G is **virtually** $\mathcal{P}$ if G has a finite index subgroup that satisfies $\mathcal{P}$.*

Example. Some examples of virtually something groups.

- Any finite group is virtually trivial.
- For any finite group F, the product $\mathbb{Z} \times F$ is virtually cyclic.
- The infinite dihedral group $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ is virtually cyclic.
- The subgroup $\mathrm{SL}_2(\mathbb{Z})$ is virtually free of rank 2. (See [Löh, 4.4] for a proof.)

Proposition. If $\mathcal{P}$ is a quasi-isometric invariant property, then a group G is virtually $\mathcal{P}$ if and only if G satisfies $\mathcal{P}$. In this case, it is redundant to say "being virtually $\mathcal{P}$".

For example, one can consider $\mathcal{P}$ = {begin growth equivalent to $x \mapsto x^2$}, or $\mathcal{P}$ = {having intermediate growth}, or $\mathcal{P}$ = {being hyperbolic} (see next section).

Nilpotent groups.

Recall that if G is a group and $A, B \subset G$, then $[A, B]$ means the subgroup of G generated by the elements $[a, b] = aba^{-1}b^{-1}$ where $a \in A$ and $b \in B$.

Definition 3.17. *Let G be a group. The* ***lower central series*** *of G is defined inductively by*

$$C_0(G) = G \quad \text{and} \quad C_{i+1}(G) = [G, C_i(G)].$$

Proposition. Let G be a group and let $i \in \mathbb{N}$. Then

1. $C_{i+1}(G)$ is a normal subgroup of both $C_i(G)$ and G.
2. The quotient $C_i(G)/C_{i+1}(G)$ is Abelian and lies in the center of $G/C_{i+1}(G)$.

That is, the *lower central series* can be written as

$$G = C_0(G) \triangleright C_1(G) \triangleright C_2(G) \triangleright \cdots \triangleright C_n(G) \triangleright \cdots .$$

Definition 3.18. *A group G is* ***nilpotent*** *(of* ***class*** *n) if there exists $n \in \mathbb{N}$ such that $C_n(G)$ is the trivial group.*
That is, the lower central series terminates at the n-th step.

Example. Some examples of nilpotent groups.

- Every abelian group G is nilpotent of class 1, since $C_1(G) = [G, G]$ is trivial.
- The subgroup G of $\mathrm{GL}_n(\mathbb{R})$ consisting of unitriangular matrices (i.e. upper triangular matrices with 1 on the diagonal) is nilpotent. One can check that $C_{n-1}(G)$ is trivial.

Recall that for an Abelian group A, the torsion-free rank $\mathrm{rk}_\mathbb{Z}(A)$ means the cardinality k such that $A = \mathbb{Z}^k \times F$ where F is a finite Abelian group.

Theorem 3.19. *Let G be a nilpotent group of class n. Then the growth of G is polynomial of degree d with*

$$d = \sum_{i=1}^{n} (i+1)\,\mathrm{rk}_\mathbb{Z}(C_i(G)/C_{i+1}(G)).$$

4 Hyperbolicity

4.1 Definitions of Gromov hyperbolic spaces

There are several definitions of hyperbolicity in the sense of Gromov. We present here three of them: those based on the *Gromov product*, *slim triangles*, and *thin triangles*. These definitions are equivalent, but they may involve different "hyperbolicity constants".

Throughout this text, the term "δ-hyperbolic" is understood in the sense of the Gromov product, unless otherwise specified.

In this section, (X, d) is a metric space.

4.1.1 Gromov product

This definition of hyperbolicity is not very intuitive, but it works for metric spaces in general. The other two definitions only apply for geodesic metric spaces. First, we need the Gromov product.

Definition 4.1. *Let $x, y, t \in X$. The* ***Gromov product*** *of x, y based at t is defined by*

$$(x, y)_t := \frac{1}{2}\big(d(x, t) + d(y, t) - d(x, y)\big).$$

Remark. The Gromov product measures the defect in the triangle inequality. This quantity has the following geometric interpretation. Let $x', y', t' \in \mathbb{R}^2$ be three points in the plane such that

$$d(x, y) = d(x', y'), \quad d(y, t) = d(y', t'), \quad \text{and} \quad d(t, x) = d(t', x').$$

Such a triangle (which may be degenerated) in $\mathbb{R}^2$ always exists and is unique up to isometry. It is called the **comparison triangle** of x, y, t. Let p' and q' be the tangent points of the incircle of the triangle $[x', y', t']$ with the sides $[x', t']$ and $[y', t']$, respectively. Then

$$(x, y)_t = d(x', p') = d(x', q').$$

One immediately observes that

$$0 \leq \langle y, z\rangle_t \leq \min\{d(x, y), d(x, z)\}.$$

Proposition. If X is a tree, then for any $x, y, z, t \in X$,

1. $(x, y)_t = d(t, [x, y])$.
2. $(x, y)_t \geq \min\{(x, z)_t, (y, z)_t\}$.

Definition 4.2. *Let $\delta \geq 0$. We say that the metric space (X, d) is* ***δ-hyperbolic*** *if, for every base point t $\in X$ and every triple $x, y, z \in X$, we have*

$$(x, y)_t \geq \min\{(x, z)_t, (y, z)_t\} - \delta.$$

We say that the metric space (X, d) is ***hyperbolic*** *if there exists $\delta \geq 0$ such that (X, d) is δ-hyperbolic.*

Informally, a δ-hyperbolic space looks like a "tree with some δ-defects". In a hyperbolic geodesic metric space, the Gromov product has the following geometric interpretation.

Proposition. If (X, d) is a δ-hyperbolic geodesic metric space, then for every x, y, z we have

$$(x, y)_z \leq d(z, [x, y]) \leq (x, y)_z + 8\delta$$

Proof. Included in Homework. □

Example. Examples of hyperbolic spaces.

1. A tree is 0-hyperbolic.
2. Every bounded space is hyperbolic, with $\delta =$ the diameter.
3. $\mathbb{H}^2$ is hyperbolic, because triangles are slim (see next section).

Example. The plane $\mathbb{R}^2$ is not hyperbolic: for every $n > 0$ one can find four points x_n, y_n, z_n, t_n such that

$$(x_n, y_n)_{t_n} < \min\{(x_n, z_n)_{t_n}, (y_n, z_n)_{t_n}\} - n.$$

Proposition. The space (X, d) is δ-hyperbolic if and only if for all $x, y, z, t \in X$, one has

$$d(x, z) + d(y, t) \leq \max\{d(x, y) + d(z, t),\ d(x, t) + d(y, z)\} + 2\delta.$$

Proof. **(⇒)** Assume (X, d) is δ-hyperbolic. Then by the Gromov product definition,

$$d(y, t) + d(z, t) - d(y, z) \geq \min\{d(y, t) + d(x, t) - d(y, x),\ d(x, t) + d(z, t) - d(x, z)\} - 2\delta.$$

We distinguish two cases.
If the minimum is the first term, then

$$d(y, t) + d(z, t) - d(y, z) \geq d(y, t) + d(x, t) - d(y, x) - 2\delta,$$

which simplifies to

$$d(x, z) + d(y, t) \leq d(x, t) + d(y, z) + 2\delta.$$

If the minimum is the second term, then

$$d(y, t) + d(z, t) - d(y, z) \geq d(x, t) + d(z, t) - d(x, z) - 2\delta,$$

which simplifies to

$$d(x, z) + d(y, t) \leq d(x, y) + d(z, t) + 2\delta.$$

Combining both cases,

$$d(x, z) + d(y, t) \leq \max\{d(x, y) + d(z, t),\ d(x, t) + d(y, z)\} + 2\delta.$$

(⇐) Exercise.

□

4.1.2 Slim triangles

This definition is used in most literature since it's more intuitive and a lot easier to describe. In the following we assume that (X, d) is a *geodesic metric space*. Recall that in this case, given two points $x, y \in X$ there exists a geodesic joining them, but the geodesic may not be unique.

A **geodesic triangle** is the data of three points $x, y, z \in X$ together with three geodesics $[x, y]$, $[y, z]$, and $[z, x]$. It will be denoted by $[x, y, z] = [x, y] \cup [y, z] \cup [z, x]$. Note that it is an abuse of notion, since the geodesics and the triangle may not be unique given x, y, z.

Given a geodesic triangle $[x, y, z]$ in X, we may consider again its comparison triangle $[x', y', z']$ in $\mathbb{R}^2$. But now we have geodesics, a point $p \in [x, y]$ can be mapped isometrically to a point $p' \in [x', y']$ in the sense that $d_X(x, p) = d_X(x', p')$ and $d_X(y, p) = d_X(y', p')$.

We also consider degenerated cases where x, y, z may not be distinct. Hence, $[x, x]$ is the point x, and $[x, y, y]$ is a "bigon" (or digon, or 2-gon). That is, two geodesics joining x and y.

Definition 4.3. *Let $\delta > 0$. A geodesic triangle $[x, y, z]$ is said to be* ***δ-slim*** *if each side is in the δ-neighborhood of the two other side. That is, we have*

$$[x, y] \subset N_\delta([y, z]) \cup N_\delta([z, x]),$$

and the two other inclusions.

Theorem 4.4. *Let (X, d) be a geodesic metric space.*

1. *If (X, d) is δ-hyperbolic, then every geodesic triangle is 4δ-slim.*
2. *If every geodesic triangle is δ-slim, then (X, d) is 3δ-hyperbolic.*

In particular, a geodesic metric space is hyperbolic if and only if its geodesic triangles are slim.

Proof. Homework. □

4.1.3 Thin triangles

This one is useful in several contexts. See next subsection. Let us first define a *comparison tripod*.

A **tripod** is a metric space obtained by gluing three intervals $[0, a]$, $[0, b]$, $[0, c]$ of $\mathbb{R}$ at their 0-points. The common point is called the *center*, and a, b, c are the three *endpoints*. Distances are given by path length. In particular, if two points lie on different branches, their distance is the sum of their distances to the center.

Let $x, y, z \in X$. A **comparison tripod** for the triple (x, y, z) is a tripod T where the three endpoints $x', y', z' \in T$ satisfy

$$d_T(x', y') = d(x, y), \quad d_T(y', z') = d(y, z), \quad d_T(z', x') = d(z, x).$$

Equivalently, if o denotes the center of the tripod, then

$$d_T(o, x') = \frac{1}{2}\big(d(x, y) + d(x, z) - d(y, z)\big) = (y, z)_x,$$

and similarly for y' and z' by cyclic permutation.

There is then a canonical projection

$$\pi \colon [x, y, z] \to T$$

that sends each side of the geodesic triangle $[x, y, z]$ isometrically onto the corresponding pair of branches of the tripod, matching x, y, z with x', y', z'.

Definition 4.5. *Let $\delta \geq 0$. Let $[x, y, z]$ be a geodesic triangle in (X, d), and let $\pi : [x, y, z] \to T$ be the canonical projection onto its comparison tripod. We say that the triangle $[x, y, z]$ is **δ-thin** if, whenever two points $p, q \in [x, y, z]$ have the same image under π, one has*

$$d(p, q) \leq \delta.$$

Theorem 4.6. *Let (X, d) be a geodesic metric space.*

1. *Every δ-thin geodesic triangle is δ-slim.*
2. *Every δ-slim geodesic triangle is 6δ-thin.*

In particular, a geodesic metric space is hyperbolic if and only if its geodesic triangles are thin.

Proof. Assertion 1. is obvious. To prove 2., we need the following lemme.

Lemma 4.7. *Let $[x, y, z]$ be a geodesic triangle and let $p \in [x, y]$, $q \in [x, z]$ such that $d(x, p) = d(x, q)$. If p is δ-close to $[x, z]$, then $d(p, q) \leq 2\delta$.*

Proof of the Lemma. Let $t \in [x, z]$ such that $d(p, t) \leq \delta$. By triangle inequality,

$$d(x, p) - d(p, t) \leq d(x, t) \leq d(x, p) + d(p, t).$$

So $|d(x, t) - d(x, p)| \leq \delta$. Since $d(x, p) = d(x, q)$ and x, t, q lie on the same geodesic with x not being the center point, we have

$$d(t, q) = |d(x, t) - d(t, q)| \leq \delta.$$

Hence, by triangle inequality,

$$d(p, q) \leq d(p, t) + d(t, q) \leq 2\delta.$$

□

Now we can prove the Theorem. Let $[x, y, z]$ be a slim triangle and let $\pi : [x, y, z] \to T$ be the canonical projection of the triangle on its comparison tripod. Let p, q, r be the preimages of the center point on $[x, y]$, $[x, z]$, $[y, z]$, respectively.

Step 1. First, we show that p, q, r are 4δ-close to each other. We estimate $d(p, q)$, the other two are symmetric.

If p is δ-close to $[x, z]$ or q is δ-close to $[x, y]$, then by the Lemma $d(p, q) \leq 2\delta$. Otherwise, because the triangle is slim, p and q are both δ-close to $[y, z]$. By the Lemma, $d(p, r) \leq 2\delta$ and $d(q, r) \leq 2\delta$. Hence $d(p, q) \leq 4\delta$ by triangle inequality.

Step 2. Now let $t \in [x, p]$ and $s \in [x, q]$ such that $d(x, t) = d(x, s)$. We estimate $d(s, t)$, the other 5 cases are similar.

If p is δ-close to $[x, z]$, then by the Lemma $d(p, q) \leq 2\delta$. Otherwise, by the slimness of the triangle, t is δ-close to $[y, z]$. By the Lemma, if t' is the point on $[r, z]$ such that $d(y, t) = d(y, t')$, then $d(t, t') \leq 2\delta$. By triangle inequality,

$$d(x, z) \leq d(x, t) + d(t, t') + d(t', z) \leq d(x, p) - d(p, t) + 2\delta + d(r, z) - d(r, t').$$

Note that by definition, $d(p, t) = d(r, t') = d(q, s)$, and $d(x, z) = d(x, p) + d(r, z)$. So $d(p, t) \leq \delta$.

By triangle inequality and Step 1.,

$$d(s, t) \leq d(s, q) + d(q, p) + d(p, t) \leq \delta + 4\delta + \delta \leq 6\delta.$$

□

4.2 The hyperbolic plane $\mathbb{H}^2$

The **hyperbolic geometry** is a negatively curved non-Euclidean geometry that emerged in the 19th century through the work of Gauss, Bolyai, and Lobachevsky.

In this section, we will state some facts about the hyperbolic plane $\mathbb{H}^2$ without proof. See [Druţu-Kapovich, Chapter 4] for details.

4.2.1 Models of $\mathbb{H}^2$

There are several models of $\mathbb{H}^2$, which are all equivalent. We discuss here the *hyperboloid model*, the *upper half plane model*, and the *Poincaré disk model*.

The hyperboloid

Definition 4.8. *The hyperboloid model of the hyperbolic plane is the set*

$$\mathbb{H}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = -1\},$$

with the Riemannian structure induced from $\mathbb{R}^3$*. That is,*

$$ds^2 = dx^2 + dy^2 + dz^2.$$

The length of a path $\gamma : [0, 1] \to \mathbb{H}^2$ described by coordinates $\gamma(t) = (x(t), y(t), z(t))$ is then

$$L(\gamma) = \int_0^1 \sqrt{dx^2 + dy^2 + dz^2} = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \int_0^1 \|\gamma'(t)\|_{\mathbb{R}^3} dt.$$

And the metric in $\mathbb{H}^2$ is defined by

$$d(a, b) = \inf\{L(\gamma) \mid \gamma : [0, 1] \to \mathbb{H}^2, \gamma(0) = a, \gamma(1) = b\}.$$

Proposition. A bi-infinite path $\gamma\colon \mathbb{R} \to \mathbb{H}^2$ is a geodesic line in $\mathbb{H}^2$ if and only if it is the intersection of $\mathbb{H}^2$ with a two dimensional linear subspace of $\mathbb{R}^3$.

Remark. Compare this model with the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ defined by

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

The path lengths and the metric can be defined similarly, and the Proposition above is intuitive.

The upper half plane

Definition 4.9. *The* ***upper half-plane model*** *of the hyperbolic plane is the set*

$$\mathbf{U} := \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$$

with the Riemannian structure

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

The length of a path $\gamma : [0,1] \to \mathbf{U}$ described by the coordinates $\gamma(t) = (x(t), y(t))$ is

$$L(\gamma) := \int_0^1 \sqrt{\frac{x'(t)^2 + y'(t)^2}{y(t)^2}} dt.$$

The metric in $\mathbf{U}$ is defined by

$$d(a,b) = \inf\{L(\gamma) \mid \gamma : [0,1] \to \mathbf{U}, \gamma(0) = a, \gamma(1) = b\}.$$

Proposition. A bi-infinite path $\gamma \colon \mathbb{R} \to \mathbf{U}$ is a geodesic line in $\mathbf{U}$ if and only if it is either a vertical semi-line or a semi-circle perpendicular to the x-axis.

The Poincaré disk

Definition 4.10. *The Poincaré disk model of the hyperbolic plane is the set*

$$\mathbf{D} := \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$$

with the Riemannian structure

$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}.$$

Exercise. Describe the length of a path and the metric in this model.

Proposition. A bi-infinite path $\gamma \colon \mathbb{R} \to \mathbf{U}$ is a geodesic line in $\mathbf{D}$ if and only if it is an arc perpendicular to the unit circle.

4.2.2 Properties of $\mathbb{H}^2$

Theorem 4.11. *The three models above define isometric metric spaces.*
We will use the notation $\mathbb{H}^2$ for all of them.

Proposition. The topology of $\mathbb{H}^2$ induced by its metric is homeomorphic to $\mathbb{R}^2$.

The following proposition motivates the introduction of this geometry.

Proposition (Parallel postulate)**.** Let L be a geodesic and P a point in $\mathbb{H}^2$. Then there are infinitely many geodesics in $\mathbb{H}^2$ passing through P that do not intersect L.

In the usual Euclidean plane, "infinitely many geodesics" is replaced by "a unique geodesic"; on the sphere $\mathbb{S}^2$, it is replaced by "no geodesic".

Proposition (Slim triangles)**.** There exists $\delta > 0$ such that geodesic triangles in $\mathbb{H}^2$ are δ-slim. That is, for any triple $a, b, c \in \mathbb{H}^2$,

$$[a,b] \subset N_\delta([b,c]) \cup N_\delta([c,a]),$$

where $[a,b]$ denotes the unique geodesic from a to b and N_δ denotes the δ-neighborhood.

4.3 Properties of Gromov hyperbolic spaces

We discuss here the notion of *quasi-geodesics* and state some properties of Gromov hyperbolic spaces without proofs, except for the QI-invariance of hyperbolicity using the stability of quasi-geodesics. See [Löh, Chapter 7], [Druţu-Kapovich, Chaper 11] or [Bridson-Haefliger, Chapter H] if you are interested.

4.3.1 Quasi-isometric invariant

Definition 4.12 (Quasi-geodesic)**.** *A* ***quasi-geodesic*** *of a metric space X is a map γ from an interval $I \subset \mathbb{R}$ to X which is a quasi-isometric embedding.*
That is, there exists $a \geq 1, c \geq 0$ such that every pair of points $x, y \in I$ satisfies

$$\frac{1}{a}|x-y| - c \leq d(\gamma(x), \gamma(y)) \leq a|x-y| + c.$$

In this case, γ is called a (a, c)-quasi-geodesic.

Every geodesic is of course a $(1, 0)$-quasi-geodesic. In general, a quasi-geodesic need not be continuous. See the following examples.

Example. Examples of quasi-geodesics.

1. The map $t \mapsto (t, \lfloor t \rfloor)$ from $\mathbb{R}$ to $\mathbb{R}^2$ is a $(1, 1)$-quasi-geodesic.
2. The map $t \mapsto (t, \sin t)$ from $\mathbb{R}$ to $\mathbb{R}^2$ is a quasi-geodesic.

Theorem 4.13 (Stability of quasi-geodesics)**.** *Let (X, d) be a δ-hyperbolic geodesic metric space. Let $\gamma : [0, \ell] \to X$ be a (a, c)-quasi-geodesic. Let γ' be any geodesic from $\gamma(0)$ to $\gamma(\ell)$, then there exists $k = k(a, c)$ such that γ and γ' are k-close to each other.*

Proof. See [Löh] Chapter 7.2. □

Remark. This is not true for metric spaces in general.
For example, in $\mathbb{R}^2$, a semi-circle of radius r is a $(\pi/2, 0)$-quasi geodesic, but the distance between the arc and the geodesic joining the endpoints of the arc is r.

Gromov hyperbolicity is a quasi-isometric invariant property. That is, we have the following theorem.

Theorem 4.14 (Quasi-isometric invariant)**.** *If X is a hyperbolic space and there is a quasi-isometric embedding from Y to X, then Y is also hyperbolic.*

Proof. We will apply the stability of quasi-geodesics and the slim-triangle criterion.

Let δ be the hyperbolicity constant of X, so that every geodesic triangle is 4δ-slim. Let $f : Y \to X$ be a (a, c)-quasi-isometric embedding. Let $[x, y, z]$ be a geodesic triangle in Y and let $p \in [x, y]$. We shall prove that p is close to the union of the two other sides.

By definition, the restriction of f on $[x, y]$ is a (a, c)-quasi-geodesic in X. By the stability of quasi-geodesics, there exists $p' \in X$ in a geodesic $[f(x), f(y)]$ such that

$$d(f(p), p') \leq k(a, c)$$

where $k(a, c)$ is given by the previous theorem.

By slimness of the geodesic triangle $[f(x), f(y), f(z)]$ in X, there exists $q' \in [f(y), f(z)] \cup [f(z), f(x)]$ such that

$$d(p', q') \leq 4\delta.$$

Wlog suppose that $q' \in [f(y), f(z)]$. By the stability of quasi-geodesics, there exists $q \in [y, z]$ such that $d(f(q), q') \leq k(a, c)$. By triangle inequality, we get

$$d(f(p), f(q)) \leq 2k(a, c) + 4\delta.$$

Since f is (a, c)-quasi-isometric,

$$d(p, q) \leq ad(f(p), f(q)) + ac \leq 2ak(a, c) + 4a\delta + ac.$$

□

4.3.2 Local-to-global quasi-geodesics

We have defined local-geodesics long ago, now it's time for local-quasi-geodesics.

Definition 4.15. *Let X be a geodesic metric space. A map $\gamma : I \to X$ where I be an interval of $\mathbb{R}$ is called a* ***local quasi-geodesic*** *if there exists $L > 0$, $a \geq 1, c \geq 0$ such that the restriction of γ on every subinterval of I of length at most L is a (a, c)-quasi-geodesic.*
In this case, γ is called an L-local (a, c)-quasi-geodesic.

It turns out that local quasi-geodesics are actually "global" geodesics in hyperbolic spaces.

Theorem 4.16 (local-to-global quasi-geodesics)**.** *Let (X, d) be a δ-hyperbolic geodesic metric space. Let $a \geq 1, c \geq 0$. Then, there exists $a' > a, c' \geq 0, L > 0$ such that every L-local (a, c)-quasi-geodesic is a (global) (a', c')-quasi-geodesic.*
In particular, local-geodesics in a hyperbolic space are quasi-geodesics.

Proof. [Druţu-Kapovich, Chaper 11.5], [Bridson-Haefliger, Chapter H.1] □

Note that this is not true in non-hyperbolic spaces. For example, in $\mathbb{R}^2$, a large circle is a local quasi-geodesic but not a quasi-geodesic.

4.4 Hyperbolic groups

4.4.1 Definition and examples

Because being hyperbolic is quasi-isometric invariant, the following definition makes sense.

Definition 4.17. *A finitely generated group is* ***hyperbolic*** *is its Cayley graph with respect to* any *finite generating set is hyperbolic.*

By Svarc-Milnor Lemma, we have the following characterization.

Proposition. A group is hyperbolic if and only if it acts (by isometries) properly and coboundedly on a hyperbolic geodesic metric space.

Example (Examples of hyperbolic groups)**.**

- Finitely generated free groups are hyperbolic, since their Cayley graphs are trees.
- Virtually cyclic groups are hyperbolic.
- Finite groups are hyperbolic.

- Surface groups with genus ≥ 2, that is, fundamental groups of surfaces of genus ≥ 2, are hyperbolic, according to their actions on $\mathbb{H}^2$.
- $\mathrm{SL}_2(\mathbb{Z})$ is hyperbolic, since it is virtually free of rank 2.

Example. For any $n \geq 2$, $\mathbb{Z}^n$ is *not* hyperbolic, since the Cayley graph is quasi-isometric to $\mathbb{R}^n$.

Exercise. Show that $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ is hyperbolic.

4.4.2 Some properties

In the following, we state some properties of hyperbolic groups without proofs. See [Druţu-Kapovich, Chapter 11] or [Löh, Chapter 7] for proofs of these results.

Some basic properties

Virtually cyclic hyperbolic groups, including finite groups and those who contain $\mathbb{Z}$ as a finite index subgroup, are called **elementary** hyperbolic groups. The other hyperbolic groups are called **non-elementary hyperbolic groups**.

Theorem 4.18. *A* non-elementary hyperbolic group *has the following properties:*

- *It contains a non-abelian free subgroup.*
- *It can not contain $\mathbb{Z}^2$ as a subgroup.*
- *It* can *contain a non hyperbolic subgroup.*
- *It is virtually torsion free.*
- *It has exponential growth.*

Theorem 4.19. *Every hyperbolic group has a finite presentation.*

Dehn function

Recall that given a presentation $G = \langle S \mid R \rangle$, a word $w \in W(S \cup S^{-1})$ is trivial in the group G if and only if w can be written as a product of conjugates of the relators. That is,

$$w = \prod_{i=1}^{n} a_i r_i a_i^{-1}$$

for some $n \in \mathbb{N}, r_i \in R \cup R^{-1}$, and $a_i \in G$. We define the *area* of w, denoted by $A(w)$, as the minimal of such n so that w can be written as a product of conjugates of n relators.

The **Dehn function** of a *finite* presentation $G = \langle S \mid R \rangle$ is the function $D_{S,R} : \mathbb{N} \to \mathbb{N}$ defined by

$$D_{S,R}(\ell) = \max\{A(w) \mid w =_G 1, |w| \leq \ell\}.$$

Theorem 4.20 (Quasi-isometric invariance)**.** *Let $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$ be groups with finite presentations such that* $\mathrm{Cay}(G_1, S_1)$ *and* $\mathrm{Cay}(G_2, S_2)$ *are quasi-isometric. Then, their Dehn functions are growth equivalent. That is,*

$$D_{S_1,R_1} \asymp D_{S_2,R_2}.$$

In particular, the growth type *of the Dehn function of a finitely presentable group is independent of its finite presentation. We denote it by D_G.*

Example. $D_{\mathbb{Z}^2} \asymp (n \mapsto n^2)$.

Dehn function gives a characterization of hyperbolicity.

Theorem 4.21. *An infinite group with a finite presentation is hyperbolic if and only if the Dehn function is linear.*

The word problem

Given a finite group presentation $G = \langle S \mid R \rangle$, we say that the presentation has a **solvable word problem** if there exists an algorithm that decides whether a word $w \in W(S \cup S^{-1})$ is trivial in G.

Theorem 4.22. *A hyperbolic group (with any finite presentation) has a solvable word problem.*

To see why this is interesting,

Theorem 4.23 (Boone-Rogers 1966)**.** *There exists a finite group presentation with an unsolvable word problem.*

Almost every group is hyperbolic

We end this course by this statement in [Gromov 1987].

Theorem 4.24 (Gromov 1987, full proof by Ol'shanskii 1992)**.** *Let $m \geq 2, k \geq 1, \ell \geq 1$. Denote by $N(m, k, \ell)$ the number of group presentations of the form $\langle x_1, \dots, x_m \mid r_1, \dots, r_k \rangle$; and by $N^h(m, k, \ell)$ the number of such presentations that gives hyperbolic groups. Then*

$$\frac{N^h(m,k,\ell)}{N(m,k,\ell)} \xrightarrow[\ell \to \infty]{} 1.$$